COMPLEX CLASSICAL FIELDS AND PARTIAL WICK ROTATIONS

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Abstract. We study some examples of complex, classical, scalar fields within the new framework that we introduced in a previous work. In these particular examples, we replace the usual functional integral by a complex functional arising from *partial Wick rotation* of a quantum field. We generalize the Feynman-Kac relation to this setting, and use it to establish the spectral condition on a cylinder. We also consider positive-temperature states.

Dedicated to Arthur Strong Wightman[1]

Contents

[1]A. S. Wightman was an inspiration to all of the authors. We learned of his passing while finishing this work, which relates to topics that fascinated him.

I. Introduction

In [21] we introduced a framework for using complex classical fields to describe neutral, scalar quantum fields. In that work we replace the real functional integral by a complex functional. In this work we study in detail one particular family of examples that provide a classical interpretation for *partial Wick rotation* of a quantum field.

Complex fields arise naturally when the heat kernel of a Hamiltonian is complex, as in the case when an interaction breaks time-reversal symmetry. A simple family of examples arises when one adds a multiple of the momentum to the Hamiltonian—the case that we study in this paper.

In the usual situation for scalar bosons, the Euclidean action $\mathfrak{A} = \mathfrak{A}(\Phi)$ is real, and the Feynman-Kac density $\mathrm{e}^{-\mathfrak{A}}$ is positive. In the case that $\mathfrak{A}$ has some other nice properties, the density $\mathrm{e}^{-\mathfrak{A}}$ can be normalized to define a probability measure. However, when the classical fields Φ are complex, the action $\mathfrak{A}$ may also be complex. Consequently, the problem of integrating $\mathrm{e}^{-\mathfrak{A}}$ is more subtle (see also [12][13]).

The mathematics of complex measures on finite-dimensional spaces poses no difficulty provided the absolute value of the measure can be integrated. The situation is more complicated for measures on function spaces, such as the measures in functional integrals. Not only can the density grow in certain complex directions, but also oscillations may lead to other difficulties with normalization. Even the case of Gaussian measures is not straightforward, so one can imagine more difficulty in the study of interactions with non-quadratic actions.

In this paper we consider perturbations of a Hamiltonian H with zero ground-state energy and with a positive heat kernel. We study perturbations of the form

$$H_{\vec{v}} = H + \vec{P} \cdot \vec{v} \; . \tag{I.1}$$

The momentum $\vec{P}$ commutes with H and generates spatial translations. As we require that $|\vec{v}| < 1$, the operator $H_{\vec{v}}$ is a multiple of the Hamiltonian in a Lorentz frame boosted by velocity $\vec{v}$, in units for which the speed of light $c = 1$.

We study free fields in arbitrary dimension; in spacetime dimension two we also treat $\mathscr{P}(\varphi)_2$-interactions on the spatial circle. In §V we introduce a generalized Feynman-Kac relation to deal with the non-linear interaction in the absence of a measure. We define a functional on a sufficiently large sub-algebra of functions of the classical fields to analyze the corresponding quantum fields.

As discussed in [21], the property of *reflection positivity* plays a key role in our setting. Further insight arises from having two reflection-positive planes in the classical framework, for one then has a symmetry relating two different quantum theories.

In §VI we use this symmetry to give a proof of

$$0 \leqslant H_{\vec{v}} \; , \tag{I.2}$$

for $\mathscr{P}(\varphi)_2$-interactions on a spatial circle. This spectrum condition was conjectured in [10] and proved in [15]; see §VI for further discussion. Without appealing to Lorentz symmetry, the estimate (I.2) results in analyticity of the imaginary-time field $\varphi(t, \vec{x})$ in the spatial variable $\vec{x}$. In particular, the spectrum condition holds when the spacetime manifold is compact in the spatial directions.

In §VII and §VIII we analyze quantization for positive temperatures. We hope that the methods we develop here can be useful in a wider context. We are currently studying a second application to charged fields.

II. Quantization

We adopt the notations and conventions of our earlier work. We analyze classical fields on space-times of the general form

$$\boldsymbol{X} = X_1 \times \cdots \times X_d \; ,$$

where each factor X_i either equals $\mathbb{R}$ (the real line) or S^1 (a circle) of length ℓ_i. The classical Gaussian, neutral, scalar field Φ is an operator-valued distributions, and all classical fields commute. The classical field acts on the Fock space

$$\mathcal{E} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{E}_n \ , \quad \text{where} \quad \mathcal{E}_n = \underbrace{\mathcal{E}_1 \otimes_s \cdots \otimes_s \mathcal{E}_1}_{n \text{ factors}} \ , \tag{II.1}$$

over the Hilbert space $\mathcal{E}_1 = L_2(\boldsymbol{X}; dx)$.

II.1. **Quantization of Vectors.** Let Φ be a field on $\mathcal{E}$ for which a unitary reflection Θ is doubly reflection-positive. While Θ usually denotes time reflection, our quantization method applies to any reflection satisfying the following list of properties:

i.) $\Theta^{-1} = \Theta^* = \Theta$ on $\mathcal{E}$;
ii.) $\Theta\mathcal{E}_\pm = \mathcal{E}_\mp$, where $\mathcal{E}_\pm$ are subspaces of $\mathcal{E}$; and,
iii.) $0 \leqslant \Theta$ on $\mathcal{E}_\pm$.

The sesquilinear form

$$(A, B) \mapsto \langle A, \Theta B\rangle_{\mathcal{E}}$$

on $\mathcal{E}_+ \times \mathcal{E}_+$ (or on $\mathcal{E}_- \times \mathcal{E}_-$) defines pre-Hilbert spaces $\mathcal{H}_{\pm,0}$, which are the quantizations of $\mathcal{E}_\pm$ with respect to the reflection Θ. The vectors in $\mathcal{H}_{\pm,0}$ are equivalence classes

$$\widehat{A} = A + N \in \mathcal{E}_\pm / \mathcal{N}_\pm \ ,$$

where $A \in \mathcal{E}_\pm$, and where $N \in \mathcal{N} \cap \mathcal{E}_\pm$ is an element of the null space $\mathcal{N}$ of the form (II.2). The inner products

$$\big\langle \widehat{A}, \widehat{B} \big\rangle_{\mathcal{H}_{\pm,0}} = \big\langle A, \Theta B \big\rangle_{\mathcal{E}} \ , \quad A, B \in \mathcal{E}_\pm \ , \tag{II.2}$$

defined initially on $\mathcal{H}_{\pm,0}$, extend to inner products on the Hilbert space $\mathcal{H}_\pm$, the completion of $\mathcal{H}_{\pm,0}$. As Θ is unitary, property *ii.*) in the list above ensures that the Hilbert spaces $\mathcal{H}_\pm$ are isomorphic, so for simplicity we denote both spaces as $\mathcal{H}$. The quantization map $\widehat{\ }$ is a contraction on vectors, namely

$$\|\widehat{A}\|_{\mathcal{H}} \leqslant \|A\|_{\mathcal{E}} \ .$$

II.2. **Quantization Domains.** To simplify notation in this section, we only consider reflections in the first coordinate. Consider an open subset $\mathcal{O}$ of the product space

$$\boldsymbol{X}_+ = X_{1,+} \times X_2 \times \cdots \times X_d \ ,$$

where $X_{1,+}$ equals the half-circle S^1_+ or the half-line $\mathbb{R}_+$ and $X_j = S^1$ or $\mathbb{R}$, $j = 2, \ldots, d$. Let $\mathscr{P}(\mathcal{O})$ denote the algebra of formal[2] polynomials in field operators averaged with C^∞-functions supported in $\mathcal{O}$.

[2]The formal product is replaced by the operator product as the formal expression is applied to the vacuum vector Ω_0^E.

Definition II.1. An open set $\mathcal{O} \subset \boldsymbol{X}_+$ is a *quantization domain* if the quantization map $A \mapsto \widehat{A}$ takes the linear subspace $\mathcal{D}(\mathcal{O}) = \mathcal{P}(\mathcal{O})\Omega_0^E$ onto a subspace $\widehat{\mathcal{D}(\mathcal{O})}$ that is dense in $\mathcal{H}$.

Remark II.2. The Reeh-Schlieder theorem of Wightman quantum field theory says that products of Minkowski space fields, smeared with test functions supported in an arbitrary open bounded spacetime region and applied to the vacuum vector, form a total set of vectors in $\mathcal{H}$. One can think of a quantization domain $\mathcal{O} \subset \boldsymbol{X}_+$ as a *classical* version of this property.

Proposition II.3 (Non-trivial Quantization Domains [18]). *Consider a covariant classical scalar field $\Phi(f)$ on $\mathcal{E}(\boldsymbol{X})$ with $X_{1,+} = \mathbb{R}_+$, which satisfies*

$$\Theta\,\Phi(x)\,\Theta = \Phi(\vartheta x)^* \ .$$

Assume that:

i.) The characteristic function $S(f) = \langle \Omega_0^E, \mathrm{e}^{i\Phi(f)}\Omega_0^E\rangle_{\mathcal{E}}$ is invariant under the action of the spacetime translation group and the time reflection on the test functions; and,

ii.) There is a constant $M < \infty$ and a Schwartz-space norm $\|\cdot\|_\alpha$ on time-zero test functions such that the following estimates hold:

$$0 \leqslant H\ , \quad \pm|\vec{P}| \leqslant M(H + \mathbb{1})\ ,$$

and

$$\pm\varphi(h) \leqslant M\|h\|_\alpha\,(H+\mathbb{1})\ .$$

Then any open set $\mathcal{O} \subset \boldsymbol{X}_+$ is a quantization domain.

II.3. **Quantization of Operators.** Consider a linear transformation T whose domain is a quantization domain $\mathcal{D}(T) \subset \mathcal{E}$. If T maps $\mathcal{E}_+\cap\mathcal{D}(T)$ into $\mathcal{E}_+$ and T maps $\mathcal{N}_+$ into $\mathcal{N}_+$, then T has a quantization $\widehat{T}_+$ on $\mathcal{H}_+$ with domain $\mathcal{D}(\widehat{T}_+) = (\mathcal{E}_+ \cap \mathcal{D}(T))^\wedge$. Explicitly,

$$\widehat{T}_+\,\widehat{A} = \widehat{TA} \quad \text{for} \quad A \in \mathcal{D}(T) \cap \mathcal{E}_+\ .$$

If in addition, T extends to a densely-defined operator on $\mathcal{E}$ with adjoint T^*, let

$$T^+ = \Theta T^* \Theta\ .$$

Assume that T^+ leaves $\mathcal{E}_+$ invariant, that is, $T^+\colon \mathcal{E}_+ \cap \mathcal{D}(T^+) \to \mathcal{E}_+$. In this case, a Schwarz inequality in $\mathcal{E}$ shows that T maps $\mathcal{N}_+$ into $\mathcal{N}_+$. In addition the adjoint of $\widehat{T}_+$ on $\mathcal{H}_+$ extends $\widehat{T^+}$. The latter denotes the quantizations of T^+ on $\mathcal{H}_+$. Similarly, one has a quantization T^- of T in case that T maps $\mathcal{E}_- \cap \mathcal{D}(T)$ into $\mathcal{E}_-$ and T maps $\mathcal{N}_-$ into $\mathcal{N}_-$.

II.4. **The Heat Kernel Semigroups.** In case $X_1 = \mathbb{R}$, let $T(t)$ denote a unitary translation group on $\mathcal{E}$, implementing translations of the distinguished *time* coordinate. Note that $T(t)^+ = T(t)$, so $\widehat{T(t)}$ is self-adjoint. Thus, $\widehat{T(t)}$ gives a self-adjoint quantization of the positive-time semigroup $T(t)$ on $\mathcal{H}_+$ and of the negative time semigroup on $\mathcal{H}_-$. In particular, the generators of these semi-groups are the (positive, self-adjoint) Osterwalder-Schrader Hamiltonians $0 \leqslant H_\pm = H_\pm^*$, and

$$\widehat{T(t)} = \begin{cases} e^{-tH_+} & \text{on } \mathcal{H}_+ \text{ if } 0 \leqslant t \, , \\ e^{tH_-} & \text{on } \mathcal{H}_- \text{ if } t \leqslant 0 \, . \end{cases}$$

We return to the case $X_1 = S^1$ in §VII.

III. Classical Gaussian Fields on $\mathbb{R}^d$

We begin by discussing the Euclidean version of the one-particle space in quantum theory on $\mathbb{R}^{d-1}$ that corresponds to the free field theory with Hamiltonian $H_{\vec{v}}$ given in (I.1). For the free field the map $H \mapsto H_{\pm\vec{v}} = H \pm \vec{P} \cdot \vec{v}$ replaces the one-particle Hamiltonian μ acting on the one-particle subspace $\mathcal{H}_1$ by

$$\mu_\pm = \mu \pm \vec{p} \cdot \vec{v} \, . \tag{III.1}$$

Here $\vec{p} = -i\nabla_{\vec{x}}$ and $\vec{v} \in \mathbb{R}^{d-1}$ is a given constant vector of length less than one. Write

$$\vec{v} = \vec{n} \tanh \beta \quad \text{and} \quad \delta = \vec{p} \cdot \vec{v} \, , \tag{III.2}$$

where $\vec{n} \in \mathbb{R}^{d-1}$ is a unit vector and $\beta \in \mathbb{R}$. Throughout this work, we assume that $m > 0$, so $|\delta| < \mu \tanh |\beta|$. As $1 - \tanh^2 \beta = \cosh^{-2} \beta = (1 - \vec{v}^{\,2})^{-1}$, we infer that $\mu^2 / \cosh^2 \beta \leqslant \mu^2 - \delta^2$. Thus,

$$0 < m\sqrt{1 - \vec{v}^{\,2}} \leqslant \mu\sqrt{1 - \vec{v}^{\,2}} \leqslant \mu_\pm \, . \tag{III.3}$$

For $\vec{v} \neq 0$, the two-point function $D_{\vec{v}}$ is complex, rather than real and positive. Nevertheless, the hermitian part of the associated heat kernel is strictly positive. Furthermore, $D_{\vec{v}}$ has two reflection planes (reflection in the time-axis and in the $\vec{v}$-axis) that are reflection-positive. The corresponding configuration space is given by $\boldsymbol{X} = \mathbb{R}^d$. For these reasons, this example fits into the framework introduced in [21].

For a non-interaction system, we obtain all information from the Gaussian expectation of classical fields. The expectation of the product of two fields defines the classical two-point function $D_{\vec{v}}$:

$$D_{\vec{v}}((s_1, \vec{x}_1), (s_2, \vec{x}_2)) = \langle \varphi(0, \vec{x}_1)\Omega_0, e^{-(s_1 - s_2)(H + \vec{P} \cdot \vec{v})} \varphi(0, \vec{x}_2)\Omega_0 \rangle \, .$$

III.1. **The Two-Point Function** $D_{\vec{v}}$**.** Let $x = (t, \vec{x}) \in \mathbb{R}^d$ and, in Fourier space, let $k = (E, \vec{k}) \in \mathbb{R}^d$. The introduction of a covariance operator $D_{\vec{v}}$ on $\mathcal{E}_1$ corresponds to the substitution $\mu \mapsto \mu_+ = \mu + \delta$ on the one-particle space. To simplify notation, denote the multiplication operators in Fourier space by

$$\mathcal{F}\mu\mathcal{F}^* = (\vec{k}^2 + m^2)^{1/2} \quad \text{and} \quad \mathcal{F}\,\vec{p}\,\mathcal{F}^* = \vec{k}\ ,$$

where $\mathcal{F}$ denotes Fourier transformation. Therefore, in Fourier space, $\mathcal{F}\delta\mathcal{F}^* = \vec{k}\cdot\vec{v}$. Consider the substitution

$$(2\pi)^{d/2}\widetilde{C}(k) = \frac{1}{E^2+\mu^2} \ \mapsto\ \frac{1}{(E+i\delta)^2+\mu^2} = (2\pi)^{d/2}\widetilde{D}_{\vec{v}}(k)\ . \quad \text{(III.4)}$$

The corresponding operator $D_{\vec{v}}$ on $L_2(\mathbb{R}^d; dx)$ has the integral kernel

$$D_{\vec{v}}(x-x') = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{1}{(E+i\delta)^2+\mu^2}\,\mathrm{e}^{ik\cdot(x-x')}\,dk\ . \quad \text{(III.5)}$$

Proposition III.1 (**Elementary Properties of** $\mathbf{D}_{\vec{v}}$)**.** *With the above conventions:*

i.) The integral kernel of $D_{\vec{v}}$ *has the representation*

$$D_{\vec{v}}(x-x') = \frac{1}{(2\pi)^{(d-1)}}\int_{\mathbb{R}^{d-1}} \mathrm{e}^{-|t-t'|\mu+(t-t')\delta+i\vec{k}(\vec{x}-\vec{x}')}\,\frac{d\vec{k}}{2\mu} \quad \text{(III.6)}$$

with δ *given in* (III.2)*;*

ii.) The decomposition of $D_{\vec{v}} = K_{\vec{v}} + iL_{\vec{v}}$ *into hermitian and skew-hermitian parts on* $L_2(\mathbb{R}^d; dx)$ *yields two hermitian operators* $K_{\vec{v}}$ *and* $L_{\vec{v}}$ *with real-valued and symmetric kernels. If* $0 < m$*, then* $0 < K_{\vec{v}}$*. Thus,*

$$\begin{aligned} D_{\vec{v}} &= D_{\vec{v}}^{\mathrm{T}} \\ 0 < K_{\vec{v}} &= K_{\vec{v}}^* = K_{\vec{v}}^{\mathrm{T}} = \overline{K_{\vec{v}}} \\ L_{\vec{v}} &= L_{\vec{v}}^* = L_{\vec{v}}^{\mathrm{T}} = \overline{L_{\vec{v}}}\ ; \textit{and,} \end{aligned}$$

iii.) Let ϑ *denote time inversion acting as a unitary on* $L_2(\mathbb{R}^d; dx)$*, and let* $\pi_{\vec{n}}$ *denote the unitary on* $L_2(\mathbb{R}^d; dx)$ *implementing reflection in the spatial plane normal to* $\vec{n}$*. The operators* $\vartheta D_{\vec{v}}$*,* $D_{\vec{v}}\vartheta$*,* $\pi_{\vec{n}}D_{\vec{v}}$*, and* $D_{\vec{v}}\pi_{\vec{n}}$ *are self-adjoint on* $L_2(\mathbb{R}^d; dx)$*.*

Proof. The first statement is a consequence of the Cauchy Integral Theorem:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{iEt}}{(E+i\delta)^2+\mu^2}\,dE = \begin{cases} \frac{\mathrm{e}^{-t(\mu-\delta)}}{2\mu} & \text{if } t>0 \\ \frac{\mathrm{e}^{t(\mu+\delta)}}{2\mu} & \text{if } t<0 \end{cases} = \frac{\mathrm{e}^{-|t|\mu+t\delta}}{2\mu}\ .$$

The second statement is a consequence of the properties of $\widetilde{D}_{\vec{v}}(k)$ in Fourier space. If $k \mapsto -k$, then $E \mapsto -E$, $\delta \mapsto -\delta$, and $\mu \mapsto \mu$. Therefore, $\widetilde{D}_{\vec{v}}(-k) = \widetilde{D}_{\vec{v}}(k)$, and consequently $D_{\vec{v}} = D_{\vec{v}}^{\mathrm{T}}$, $K_{\vec{v}} = K_{\vec{v}}^{\mathrm{T}}$ and $L_{\vec{v}} = L_{\vec{v}}^{\mathrm{T}}$ are symmetric, as claimed. Both $K_{\vec{v}}$ and $L_{\vec{v}}$ act as multiplication operators by real-valued functions in Fourier space, so they are hermitian and real. Their explicit form is

$$(2\pi)^{d/2}\,\widetilde{K}_{\vec{v}}(k) = \frac{E^2+\mu^2-\delta^2}{(E^2+(\mu-\delta)^2)\,(E^2+(\mu+\delta)^2)}\ ,$$

and

$$(2\pi)^{d/2}\,\widetilde{L}_{\vec{v}}(k) = \frac{-2E\delta}{(E^2+(\mu-\delta)^2)\,(E^2+(\mu+\delta)^2)}\ . \tag{III.7}$$

The bound (III.3) shows $\widetilde{K}_{\vec{v}}(k)$ is non-vanishing as long as $m \neq 0$. Thus, $0 < K_{\vec{v}}$, and $K_{\vec{v}}$ is invertible. Also,

$$\frac{\widetilde{L}_{\vec{v}}(k)}{\widetilde{K}_{\vec{v}}(k)} = \frac{-2E\delta}{E^2+\mu^2-\delta^2}\ . \tag{III.8}$$

Finally, consider the third statement. Time reflection in Fourier space leaves μ, δ, and $\vec{k}$ invariant and sends $E \mapsto -E$. Thus, under time inversion,

$$\widetilde{D}_{\vec{v}}(k) \mapsto \overline{\widetilde{D}_{\vec{v}}(k)}\ .$$

In configuration space this implies that $\vartheta D_{\vec{v}} \vartheta = D_{\vec{v}}^*$. Hence $\vartheta D_{\vec{v}}$ and $D_{\vec{v}}\vartheta$ are self-adjoint.

In Fourier space, spatial-reflection $\pi_{\vec{n}}$ acts as follows: it leaves E invariant, and it sends $\vec{k} \mapsto \pi_{\vec{n}}\vec{k} = \vec{k} - 2(\vec{n}\cdot\vec{k})\vec{n}$. This is a consequence of

$$\begin{aligned}\pi_{\vec{n}}\,\vec{x}\cdot\vec{k} &= (\vec{x}-2(\vec{x}\cdot\vec{n})\vec{n})\cdot\vec{k} = \vec{k}\cdot\vec{x} - 2(\vec{k}\cdot\vec{n})(\vec{x}\cdot\vec{n}) \\ &= \vec{x}\cdot(\vec{k}-2(\vec{k}\cdot\vec{n})\vec{n})\ .\end{aligned}$$

Under this transformation, $\mu \mapsto \mu$ and $\delta \mapsto -\delta$. Thus, $\widetilde{D}_{\vec{v}}(k) \mapsto \overline{\widetilde{D}_{\vec{v}}(k)}$, and in configuration space $\pi_{\vec{n}}\, D_{\vec{v}}\, \pi_{\vec{n}} = D_{\vec{v}}^*$. Thus, $\pi_{\vec{n}} D_{\vec{v}}$ and $D_{\vec{v}}\, \pi_{\vec{n}}$ are self-adjoint. □

We now provide bounds on the self-adjoint real and imaginary parts of $D_{\vec{v}} = K_{\vec{v}} + iL_{\vec{v}}$. Denote the absolute value by $|D_{\vec{v}}| = (D_{\vec{v}}^*\, D_{\vec{v}})^{1/2}$.

Proposition III.2. *The operators $K_{\vec{v}}$, $L_{\vec{v}}$, $D_{\vec{v}}$, and C on $L_2(\mathbb{R}^d; dx)$ mutually commute and satisfy*

$$K_{\vec{v}} \leqslant |D_{\vec{v}}| \leqslant (\cosh\beta) K_{\vec{v}}\ .$$

Moreover,

$$\left(\frac{1}{2\cosh^2\beta}\right) C < K_{\vec{v}} < (\cosh^4\beta) C \ , \textit{and} \ \sup_k \left|\frac{\widetilde{L}_{\vec{v}}(k)}{\widetilde{K}_{\vec{v}}(k)}\right| = \sinh|\beta| \ , \quad \text{(III.9)}$$

as well as,

$$(2\cosh^2\beta)^{-1} C < |D_{\vec{v}}| < (\cosh^5\beta) C \ . \quad \text{(III.10)}$$

Proof. The operators $K_{\vec{v}}$, $L_{\vec{v}}$, $D_{\vec{v}}$, and C are all translation-invariant, so they commute. Furthermore, $\|K_{\vec{v}}\| = (2\pi)^{d/2}\sup_k |\widetilde{K}_{\vec{v}}(k)|$. Note that

$$\begin{aligned}(E^2+(\mu-\delta)^2)(E^2+(\mu+\delta)^2) &= E^4 + 2E^2(\mu^2+\delta^2) + (\mu^2-\delta^2)^2 \\ &< E^4 + 4E^2\mu^2 + \mu^4 \\ &< 2(E^2+\mu^2)^2 \ .\end{aligned}$$

From (III.3) and (III.7) one then infers the lower bound for $K_{\vec{v}}$ in (III.9). To establish the upper bound on $K_{\vec{v}}$, use

$$\begin{aligned}E^4 + 2E^2(\mu^2+\delta^2) + (\mu^2-\delta^2)^2 &> E^4 + 2E^2\mu^2 + \frac{\mu^4}{\cosh^4\beta} \\ &\geqslant \left(\frac{E^2+\mu^2}{\cosh^2\beta}\right)^2 ,\end{aligned}$$

which entails

$$\begin{aligned}(2\pi)^{d/2}\,\widetilde{K}_{\vec{v}}(k) &< \cosh^4\beta\,\frac{(E^2+\mu^2-\delta^2)}{(E^2+\mu^2)^2} \\ &\leqslant (2\pi)^{d/2}\,(\cosh^4\beta)\widetilde{C}(k) \ .\end{aligned}$$

The upper bound on $K_{\vec{v}}$ then follows. We also use the explicit forms in (III.7) to bound the ratio (III.8). From (III.3) we conclude that

$$\begin{aligned}\left|\frac{\widetilde{L}_{\vec{v}}(k)}{\widetilde{K}_{\vec{v}}(k)}\right| &= \frac{2|E\,\vec{k}\cdot\vec{n}|\frac{1}{\cosh\beta}}{E^2+\mu^2-\delta^2}\,\sinh|\beta| \\ &\leqslant \frac{E^2+\frac{\vec{k}^2}{\cosh^2\beta}}{E^2+\mu^2-\delta^2}\,\sinh|\beta| \\ &\leqslant \frac{E^2+\frac{\vec{k}^2}{\cosh^2\beta}}{E^2+\frac{\mu^2}{\cosh^2\beta}}\,\sinh|\beta| < \sinh|\beta| \ . \quad \text{(III.11)}\end{aligned}$$

In fact, one can approach the bound (III.11) by choosing

$$\vec{k}\cdot\vec{n} = |\vec{k}| = -E\cosh\beta > 0 \ .$$

Then

$$\frac{\widetilde{L}_{\vec{v}}(k)}{\widetilde{K}_{\vec{v}}(k)} = \frac{2E^2 \sinh\beta}{2E^2 + m^2} \to \sinh\beta \quad \text{as} \quad E \to \infty\ .$$

This shows that the upper bound in (III.11) is the best possible, so the equality in (III.9) holds.

Finally, we bound $|D_{\vec{v}}| = (K_{\vec{v}}^2 + L_{\vec{v}}^2)^{1/2}$. Since $K_{\vec{v}}$ and $L_{\vec{v}}$ are self-adjoint and commute, the bound (III.9) yields

$$\begin{aligned} K_{\vec{v}} &\leqslant |D_{\vec{v}}| \\ &= (\mathbb{1} + (L_{\vec{v}} K_{\vec{v}}^{-1})^2)^{1/2} K_{\vec{v}} \\ &\leqslant (1 + \sinh^2\beta)^{1/2}\, K_{\vec{v}} \\ &= (\cosh\beta) K_{\vec{v}}\ , \end{aligned}$$

where $\mathbb{1}$ denotes the identity operator. □

III.2. **Time-Reflection Positivity.** Consider the positive-time half-space $\boldsymbol{X}_+ = \mathbb{R}_+ \times \mathbb{R}^{d-1}$; the negative-time half-space $\boldsymbol{X}_-$ is defined similarly. Let $L_{2,+} = L_2(\boldsymbol{X}_\pm; dx)$ denote the subspace of $L_2(\mathbb{R}^d; dx)$ consisting of functions supported in $\boldsymbol{X}_\pm$.

Proposition III.3. *The operators $\vartheta D_{\vec{v}}$ and $D_{\vec{v}}\,\vartheta$ have positive expectations on $L_{2,+}$. The corresponding Osterwalder-Schrader Hamiltonians μ_+ for $\vartheta D_{\vec{v}}$ and μ_- for $D_{\vec{v}}\,\vartheta$ are the Hamiltonians $\mu_\pm$ defined in* (III.1) *acting on $\mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$.*

Proof. We establish positivity of $\vartheta D_{\vec{v}}$ on $L_{2,+}$ directly from the form of its integral kernel (III.6). The operator $\vartheta D_{\vec{v}}$ is hermitian by Proposition III.5, and its integral kernel on $L_{2,+} \times L_{2,+}$ is

$$(\vartheta D_{\vec{v}})(x, x') = \frac{\mathrm{e}^{-(t+t')(\mu+\delta)}}{2\mu} (\vec{x} - \vec{x}')\ , \tag{III.12}$$

which exhibits its positivity and shows that the Osterwalder-Schrader Hamiltonian is $\mu_+ = \mu + \delta$, acting on the Sobolev space $\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$ with inner product

$$\big\langle \cdot, \cdot \big\rangle_{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})} = \Big\langle \tfrac{1}{\sqrt{2\mu}}\ \cdot\ ,\ \tfrac{1}{\sqrt{2\mu}}\ \cdot \Big\rangle_{L_2(\mathbb{R}^{d-1}; d\vec{x})}\ . \tag{III.13}$$

Let $f \in L_{2,+}$ be smooth, and consider $f_t(\vec{x}) = f(t, \vec{x})$ to be a family of functions of $\vec{x} \in \mathbb{R}^{d-1}$. It follows that

$$\langle f, \vartheta D_{\vec{v}} g \rangle_{L_2(\mathbb{R}^d)} = \Big\langle \int_0^\infty \mathrm{e}^{-t\mu_+} f_t\, dt\ , \int_0^\infty \mathrm{e}^{-t\mu_+} g_t\, dt \Big\rangle_{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})}\ .$$

The expression (III.6) shows that the integral kernel of $D_{\vec{v}}\,\vartheta$ is

$$(D_{\vec{v}}\,\vartheta)(x,x') = \frac{\mathrm{e}^{-(t+t')(\mu-\delta)}}{2\mu}(\vec{x}-\vec{x}')\;,$$

which exhibits reflection positivity and shows that the Osterwalder-Schrader Hamiltonian for the operator $D_{\vec{v}}\,\vartheta$ is $\mu_- = \mu - \delta$. In this case,

$$\langle f, D_{\vec{v}}\,\vartheta\, g\rangle_{L_2(\mathbb{R}^d)} = \left\langle \int_0^\infty \mathrm{e}^{-t\mu_-} f_t\, dt\;,\; \int_0^\infty \mathrm{e}^{-t\mu_-} g_t\, dt \right\rangle_{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})}\;.$$

□

Proposition III.4. *Let $\mathcal{K}_{+,0} \subset \mathcal{K}_+$ be the dense subset defined as the linear span of $C_0^\infty(S^1_+) \times C_0^\infty(\mathbb{R}^{d-1})$. Define the* Osterwalder-Schrader quantization maps $\wedge_\pm : \mathcal{K}_{+,0} \to \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$,

$$\widehat{f}^{\,\pm} = \int_0^\infty \mathrm{e}^{-t\mu_\pm} f_t\, dt\;. \tag{III.14}$$

Then, it follows that

$$\langle f, \vartheta D_{\vec{v}} g\rangle_{L_2(\mathbb{R}^d)} = \left\langle \widehat{f}^{\,+}, \widehat{g}^{\,+}\right\rangle_{\mathfrak{H}_{-\frac{1}{2}}} \tag{III.15}$$

$$\langle f, D_{\vec{v}}\,\vartheta\, g\rangle_{L_2(\mathbb{R}^d)} = \left\langle \widehat{f}^{\,-}, \widehat{g}^{\,-}\right\rangle_{\mathfrak{H}_{-\frac{1}{2}}}\;. \tag{III.16}$$

III.3. **The Classical Gaussian Field.** The neutral field $\Phi(x)$ acts on $\mathcal{E}$ as a sesquilinear form defined as a linear function of commuting coordinates

$$\widetilde{Q}(k) = \widetilde{Q}(-k)^*\;.$$

The latter operators are linear functions of the creation and annihilation operators on $\mathcal{E}$; see §II of [21]. We set

$$\Phi(x) = (2\pi)^{-d/2} \int \widetilde{Q}(k)\,\widetilde{\sigma}(k)\,\mathrm{e}^{ik\cdot x}\, dk\;, \tag{III.17}$$

and note that

$$\Phi(x)^* = (2\pi)^{-d/2} \int \widetilde{Q}(k)\,\overline{\widetilde{\sigma}(-k)}\,\mathrm{e}^{ik\cdot x}\, dk\;.$$

The expectations of products of such fields in Ω_0^E obey a Gaussian recursion relation,

$$S_n(f) = (n-1)S_2(f)\,S_{n-2}(f),$$

where $S_n(f) = \langle \Omega_0^E, \Phi(f)^n \Omega_0^E\rangle$. Permutation symmetry ensures that $\langle \Omega_0^E, \Phi(f_1)\cdots\Phi(f_n)\Omega_0^E\rangle$ is fixed uniquely through polarization. The expectation of the product of two fields equals the propagator $D_{\vec{v}}$, with integral kernel

$$\begin{aligned}\langle \Omega_0^E, \Phi(x)\,\Phi(x')\Omega_0^E\rangle &= D_{\vec{v}}(x-x') \\ &= (2\pi)^{-d}\int \widetilde{\sigma}(k)\,\widetilde{\sigma}(-k)\,\mathrm{e}^{ik(x-x')}\,dk \\ &= (\sigma\sigma^{\mathrm{T}})(x,x')\ .\end{aligned}$$

Furthermore, the estimate (III.10) shows that $\Phi(f)\Omega_0^E$ satisfies the bound

$$\begin{aligned}\|\Phi(f)\Omega_0^E\|^2 &= \langle \Omega_0^E, \Phi(f)^*\Phi(f)\Omega_0^E\rangle \\ &= \langle f, \overline{\sigma}\sigma^{\mathrm{T}} f\rangle \\ &= \|\sigma^{\mathrm{T}} f\|_{L_2}^2 \\ &= \||D_{\vec{v}}|^{1/2} f\|_{L_2}^2 \\ &\leqslant (\cosh^5\beta)\,\|C^{1/2}f\|_{L_2}^2\ .\end{aligned}$$

While $D_{\vec{v}}$ does not determine $\sigma_{\vec{v}}$ uniquely, an elementary solution is to define $\sigma_{\vec{v}}$ as a square root of $D_{\vec{v}}$. Proposition III.1 shows that $D_{\vec{v}}$ has a positive real part, so we can define its square root as also having a positive real part. In Fourier space, this square root depends continuously on k. Write

$$\sigma_{\vec{v}} = D_{\vec{v}}^{1/2} \tag{III.18}$$

or, in Fourier space,

$$\widetilde{\sigma}_{\vec{v}}(k) = (2\pi)^{d/4}\,\widetilde{D}_{\vec{v}}(k)^{1/2}\ . \tag{III.19}$$

In configuration space, one has for this example,

$$\begin{aligned}\sigma_{\vec{v}} &= \left((-i\tfrac{\partial}{\partial t} + \nabla_{\vec{x}}\cdot\vec{v})^2 - \nabla_{\vec{x}}^2 + m^2\right)^{-1/2} \\ &= \left(-\Delta + m^2 + (\nabla_{\vec{x}}\cdot\vec{v})^2 - 2i\tfrac{\partial}{\partial t}(\nabla_{\vec{x}}\cdot\vec{v})\right)^{-1/2}\ .\end{aligned}$$

Correspondingly, the formula for $D_{\vec{v}}$ in configuration space is

$$D_{\vec{v}} = \left(-\Delta + m^2 + (\nabla_{\vec{x}}\cdot\vec{v})^2 - 2i\tfrac{\partial}{\partial t}(\nabla_{\vec{x}}\cdot\vec{v})\right)^{-1}\ . \tag{III.20}$$

Proposition III.5 (Properties of the Classical Field). *Let $D_{\vec{v}}$ have the form presented in* (III.5)*, and let the operator $\sigma_{\vec{v}}$ be given by* (III.19)*. Then*

$$\sigma_{\vec{v}} = \sigma_{\vec{v}}^{\mathrm{T}}\ , \quad \vartheta\,\sigma_{\vec{v}}^{\mathrm{T}}\,\vartheta = \sigma_{\vec{v}}^*\quad \textit{and}\quad \pi_{\vec{v}}\,\sigma_{\vec{v}}^{\mathrm{T}}\,\pi_{\vec{v}} = \sigma_{\vec{v}}^*\ . \tag{III.21}$$

Also,

$$\vartheta D_{\vec{v}}\vartheta = D_{\vec{v}}^*\ . \tag{III.22}$$

The field Φ *transforms under time and spatial reflections as*

$$\Theta\,\Phi(x)\,\Theta = \Phi(\vartheta x)^* \quad \textit{and} \quad \Pi_{\vec{n}}\,\Phi(x)\,\Pi_{\vec{n}} = \Phi(\pi_{\vec{n}}x)^* \,. \tag{III.23}$$

The field $\Phi(x)$ *is hermitian if and only if* $\vec{v} = 0$*.*

Proof. The operator $D_{\vec{v}} = D_{\vec{v}}^{\mathrm{T}}$ is symmetric; thus, in Fourier space $\widetilde{D}_{\vec{v}}(-k) = \widetilde{D}_{\vec{v}}(k)$. The square root $\widetilde{\sigma}_{\vec{v}}(k)$ has a positive real part and is continuous in k, so it satisfies $\widetilde{\sigma}_{\vec{v}}(-k) = \widetilde{\sigma}_{\vec{v}}(k)$. Hence, the operator $\sigma_{\vec{v}}$ is also symmetric. The field $\Phi(x)$ is hermitian if $\widetilde{\sigma}(k) = \overline{\widetilde{\sigma}(-k)}$, namely if the operator σ is real. Therefore, the field $\Phi(x)$ is hermitian on $\mathcal{E}$ only in the case that $\vec{v} = 0$.

In Proposition III.1, we showed that $\vartheta D_{\vec{v}}$ is self-adjoint, so

$$\widetilde{\sigma}_{\vec{v}}(\vartheta k)^2 = \overline{\widetilde{\sigma}_{\vec{v}}(k)}^{\,2} \,.$$

As $(2\pi)^{d/2}\widetilde{D}_{\vec{v}}(k) = \widetilde{\sigma}_{\vec{v}}(k)^2$ has a positive real part, its square root with positive real part also satisfies $\widetilde{\sigma}_{\vec{v}}(\vartheta k) = \overline{\widetilde{\sigma}_{\vec{v}}(k)}$. The Fourier transform of this relation is equivalent to the second identity in (III.21). The proof of the third identity is similar. The relation (III.22) follows from (III.21) and $D_{\vec{v}} = \sigma_{\vec{v}}^2$. We have shown the equivalence of the transformation properties (III.23) and (III.21) for fields of the form (III.17) in Propositions II.1 and II.5 of [21]. □

III.4. **The Gaussian Quantum Field.** The time-zero quantum field results from the quantization

$$\varphi(\vec{x}) = \widehat{\Phi(0,\vec{x})}$$

of the time-zero classical field and acts on the Fock space

$$\mathcal{H} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_n \,, \quad \text{where} \quad \mathcal{H}_n = \underbrace{\mathcal{H}_1 \otimes_s \cdots \otimes_s \mathcal{H}_1}_{n \text{ factors}} \,,$$

with $\mathcal{H}_1 = \mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$. Recall that $\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$ is the Sobolev space defined in (III.13), which is just the usual one-particle space for the real, free scalar field. The Gaussian nature of the Fock space $\mathcal{E}$, together with the fact that $\Phi(f)$ maps $\mathcal{E}_+$ into $\mathcal{E}_+$, implies that

$$\varphi(h) = \widehat{\Phi(0,h)} = \widehat{\Phi(f)} \,, \qquad f = \delta \otimes h \,.$$

For a real test-function h, the reflection property determined by Proposition III.5 yields

$$\begin{aligned}\|\varphi(h)^n\Omega_0\|^2_{\mathcal{H}} &= \langle \Phi(f)^n\Omega_0^E, \Theta\Omega_0^E\rangle_{\mathcal{E}} \\ &= \langle \Omega_0^E, (\Theta\Phi(f)^*\Theta)^n\Phi(f)^n\Omega_0^E\rangle_{\mathcal{E}} \\ &= \langle \Omega_0^E, \Phi(f)^{2n}\Omega_0^E\rangle_{\mathcal{E}} \\ &= (2n-1)!!\langle h, h\rangle^n_{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})} .\end{aligned}$$

Expressed in terms of the creation operators for the free field $a^*(\vec{x})$ one has that, for the Fock-space zero-particle vector Ω_0,

$$\begin{aligned}\|\varphi(h)\Omega_0\|^2_{\mathcal{H}} &= \langle h, h\rangle_{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})} \\ &= \|a^*(h)\Omega_0\|^2_{\mathcal{H}} \\ &= \|a^*((2\mu)^{-1/2}h)\Omega_0\|^2_{L^2(\mathbb{R}^{d-1})} .\end{aligned}$$

This shows that the time-zero field has the same expectations as the time-zero *free* field. Furthermore, the time-zero quantum fields generate an abelian algebra.

The expectations of products of classical fields on $\mathcal{E}$ satisfy Gaussian recursion relations, so their quantizations also satisfy Gaussian recursion relations. The expectations $\langle \Omega_0, \varphi(h_1)\cdots\varphi(h_n)\Omega_0\rangle$ can be obtained from the expectations of $\langle \Omega_0, \varphi(h)^n\Omega_0\rangle$ by polarization. In this case,

$$\|\varphi(h)\Omega_0\|^2 = \langle h_1, h_1\rangle_{\mathfrak{H}_{-\frac{1}{2}}} + \langle h_2, h_2\rangle_{\mathfrak{H}_{-\frac{1}{2}}} = \langle h, h\rangle_{\mathfrak{H}_{-\frac{1}{2}}} ,$$

as the scalar product in $\mathfrak{H}_{-\frac{1}{2}}$ is hermitian. Thus, the time-zero field on $\mathcal{H}$ is hermitian and has the form

$$\varphi(\vec{x}) = (2\pi)^{-(d-1)/2}\int \Big(a(\vec{k})^* + a(-\vec{k})\Big)\, \mathrm{e}^{-i\vec{k}\cdot\vec{x}}\, d\vec{k} .$$

The Hamiltonian $H_\pm = H_{\text{free}} \pm \vec{P}\cdot\vec{v}$ acts on the n-particle subspace as a direct sum of the 1-particle Hamiltonians $\mu_\pm = \mu \pm \delta$. Recall that H_{free} and $\vec{P}$ are the free-field Hamiltonian and momentum operator on $\mathcal{H}$. Setting

$$\varphi_\pm(t, \vec{x}) = \mathrm{e}^{itH_\pm}\varphi(\vec{x})\mathrm{e}^{-itH_\pm} ,$$

we obtain Wightman functions

$$\begin{aligned}&\mathcal{W}^{(n)}\big(\Lambda_\pm^{-1}(t_1, \vec{0}) + (0, \vec{x}_1), \ldots, \Lambda_\pm^{-1}(t_n, \vec{0}) + (0, \vec{x}_n)\big) \\ &\qquad = \langle \Omega_0, \varphi_\pm(t_1, \vec{x}_1)\cdots\varphi_\pm(t_n, \vec{x}_n)\Omega_0\rangle ,\end{aligned}$$

where the Lorentz transformation $\Lambda_\pm$ gives the boost by velocity $\pm\vec{v}$.

III.5. **Spatial Reflection Positivity.** In order to establish spatial reflection positivity, we proceed as in Section III.2 but evaluate the Fourier transform in a spatial direction. Let $k = (E, \vec{k}) \in \mathbb{R}^d$ and let $\vec{k}^\perp \in \mathbb{R}^{d-2}$ denote the component of $\vec{k} \in \mathbb{R}^{d-1}$ in the subspace of dimension $d-2$ orthogonal to the vector $\vec{n}$. Also, let $\nu = \nu(\vec{n}, E, \vec{k}^\perp)$ be the positive square root

$$\nu = \left(E^2 + \tfrac{\vec{k}^{\perp\,2}+m^2}{\cosh^2\beta}\right)^{1/2} \geqslant (E^2 + \tfrac{m^2}{\cosh^2\beta})^{1/2} > |E|\;.$$

In particular, $\nu \pm E\tanh\beta > 0$. Define the one-particle Sobolev space $\widetilde{\mathfrak{H}}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$ as the Hilbert space of functions with coordinates $(E, \vec{k}^\perp) \in \mathbb{R}^{d-1}$ and with inner product

$$\big\langle \cdot, \cdot \big\rangle_{\widetilde{\mathfrak{H}}_{-\frac{1}{2}}(\mathbb{R}^{d-1})} = \left\langle \tfrac{1}{\sqrt{2\nu}}\,\cdot, \tfrac{1}{\sqrt{2\nu}}\,\cdot \right\rangle_{L_2(\mathbb{R}^{d-1}; dE\, d\vec{k}^\perp)}\;. \tag{III.24}$$

Let ν also denote the corresponding pseudo-differential operator acting in configuration space,

$$\nu = \left(-\tfrac{\partial^2}{\partial t^2} + \tfrac{-\nabla_{\vec{x}}^2 + (\vec{n}\cdot\nabla_{\vec{x}})^2 + m^2}{\cosh^2\beta}\right)^{1/2}\;.$$

Finally, let $U(s)$ denote translation in the coordinate direction.

Proposition III.6. *The operators $\pi_{\vec{n}} D_{\vec{v}}$ and $D_{\vec{v}}\pi_{\vec{n}}$ have both positive expectations on $L_2(\boldsymbol{X}_{\vec{n}+}; dE\, d\vec{k}^\perp)$. The corresponding Osterwalder-Schrader Hamiltonians ν_+ and ν_- (for $\pi_{\vec{n}} D_{\vec{v}}$ and for $D_{\vec{v}}\pi_{\vec{n}}$) both act on the Sobolev space of functions $\widetilde{\mathfrak{H}}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$ defined in* (III.24)*. In momentum space, the explicit forms of the Hamiltonians are*

$$\nu_\pm = (\cosh^2\beta)(\nu \pm E\tanh\beta)\;. \tag{III.25}$$

Remark III.7. In case $d = 2$, one has $\vec{k}^\perp = 0$, so ν is only a function of the energy, m and β. In this case,

$$\nu = \left(E^2 + \tfrac{m^2}{\cosh^2\beta}\right)^{1/2}\;,$$

and

$$\nu_\pm = \cosh^2\beta\left(\left(E^2 + \tfrac{m^2}{\cosh^2\beta}\right)^{1/2} \pm E\tanh\beta\right)\;. \tag{III.26}$$

This has a similar form to $\mu_\pm$ of (III.1), but with an overall multiple of $\cosh^2\beta$ and with a mass modified by the factor $(\cosh\beta)^{-1}$. Hence, in configuration space,

$$\nu = \left(-\tfrac{\partial^2}{\partial t^2} + \tfrac{m^2}{\cosh^2\beta}\right)^{1/2}\;,$$

and

$$\mathcal{F}^* \nu_\pm \mathcal{F} = \cosh^2\beta \left((-\tfrac{\partial^2}{\partial t^2} + \tfrac{m^2}{\cosh^2\beta})^{1/2} \pm i \tfrac{\partial}{\partial t} \tanh\beta \right) .$$

For $d > 2$, one must also scale the remaining spatial variables by $\cosh\beta$.

Proof. Under rotations, the operator $D_{\vec{v}}$ transforms by the rotation of $\vec{n}$, so it is no loss of generality to assume that $\vec{n}$ points in the direction of the coordinate x_1. In Fourier space $\delta = k_1 \tanh\beta$. Write the inverse of $(2\pi)^{d/2}\widetilde{D}_{\vec{v}}(k)$ as

$$\begin{aligned}(E+i\delta)^2 + \mu^2 &= \tfrac{1}{\cosh^2\beta}\Big(k_1^2 + (iE\sinh 2\beta)k_1 \\ &\qquad + \big(E^2 + \vec{k}^{\perp 2} + m^2\big)\cosh^2\beta\Big) \\ &= \tfrac{1}{\cosh^2\beta}(k_1 - ik_+)(k_1 - ik_-) ,\end{aligned}$$

where the roots $0 < k_+, -k_-$ are given by

$$k_\pm = (\pm\nu - E\tanh\beta)\cosh^2\beta .$$

With this information, one can evaluate the integral

$$\begin{aligned}(2\pi)^{-1}\int_{-\infty}^{\infty} \frac{e^{ik_1\xi}}{(E+i\delta)^2+\mu^2}\, dk_1 &= \frac{\cosh^2\beta}{k_+ - k_-}\begin{cases} e^{-k_+\xi} \\ e^{-k_-\xi} \end{cases} \\ &= \frac{\cosh\beta}{2\nu}\begin{cases} e^{-k_+\xi} , & \text{if } \xi > 0 \\ e^{-k_-\xi} , & \text{if } \xi < 0 .\end{cases}\end{aligned}$$

Replace ξ by $x_1 - x_1'$ and apply the reflection $\pi_{\vec{n}}$. Infer that the integral kernel $(\pi_{\vec{n}} D_{\vec{v}})(x,x')$ of $\pi_{\vec{n}} D_{\vec{v}}$, acting on functions in $L_2(\mathbb{R}^d_{\vec{n}+}; dx)$, is

$$(\pi_{\vec{n}} D_{\vec{v}})(x,x') = \left(\frac{e^{k_-(x_1+x_1')}}{2\nu}\right)(t-t', x_2 - x_2', \ldots, x_d - x_d') .$$

Thus, conclude that spatial reflection positivity holds for $\pi_{\vec{n}}\mathcal{D}_{\vec{v}}$, and that its one-particle Osterwalder-Schrader Hamiltonian is

$$\nu_+ = -k_- = (\nu + E\tanh\beta)\cosh^2\beta ,$$

which acts naturally on the one-particle Hilbert space $\widetilde{\mathfrak{H}}_{-\frac{1}{2}}$ defined in (III.24).

Repeating the same argument for $D_{\vec{v}}\,\pi_{\vec{n}}$, find that

$$(D_{\vec{v}}\,\pi_{\vec{n}})(x,x') = \left(\frac{e^{-k_+(x_1+x_1')}}{2\nu}\right)(t-t', x_2 - x_2', \ldots, x_d - x_d') .$$

This shows that reflection positivity holds for $D_{\vec{v}}\,\pi_{\vec{n}}$, and that its Osterwalder-Schrader Hamiltonian is

$$\nu_- = k_+ = (\nu - E\tanh\beta)\cosh^2\beta ,$$

which acts on the same one-particle space $\widetilde{\mathfrak{H}}_{-\frac{1}{2}}$ as ν_+. □

III.6. **Quantization of Spatial Reflection Positivity.** We use coordinates such that $x = (t, x^{\perp}_{\vec{n}}, x_{\vec{n}}) \in \mathbb{R}^d$ to introduce the $x_{\vec{n}} = 0$ quantum field $\widetilde{\varphi}(t)$ as the quantization of $\Phi(t, x^{\perp}, 0)$ with respect to the inner product determined by the matrix elements of $\Pi_{\vec{n}}$ on $\widetilde{\mathcal{E}}_+ \equiv \mathcal{E}_{\vec{n}+}$, namely,

$$\widetilde{\varphi}(t, x^{\perp}) = \Phi(\widetilde{t, x^{\perp}}, 0).$$

The quantum field $\widetilde{\varphi}(t)$ acts on the Fock space

$$\widetilde{\mathcal{H}} = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \widetilde{\mathcal{H}}_n \,, \quad \text{where} \quad \widetilde{\mathcal{H}}_n = \underbrace{\widetilde{\mathcal{H}}_1 \otimes_s \cdots \otimes_s \widetilde{\mathcal{H}}_1}_{n \text{ factors}} \,,$$

where $\widetilde{\mathcal{H}}_1 = \widetilde{\mathfrak{H}}_{-\frac{1}{2}}$ is the spatial one-particle Sobolev space defined in (III.24). The scalar $1 \in \mathbb{C}$ is the standard zero-particle state $\widetilde{\Omega}_0$ in $\widetilde{\mathcal{H}}$. Note that Π maps $\widetilde{\mathcal{E}}_+$ to $\widetilde{\mathcal{E}}_-$, and also $\pi_{\vec{n}}(g \otimes \delta) = g \otimes \delta$. Thus, for $f = g \otimes \delta$ real-valued,

$$\widetilde{\varphi}(g) = \int \widetilde{\varphi}(t, x^{\perp}) g(t, x^{\perp})\, dt\, dx^{\perp} = \widetilde{\Phi(f)}$$

and

$$\begin{aligned} \|\widetilde{\varphi}(g)^n \widetilde{\Omega}_0\|^2_{\mathcal{H}} &= \langle \Phi(f)^n \Omega_0^E, \Pi \Phi(f)^n \Omega_0^E \rangle_{\mathcal{E}} \\ &= \langle \Omega_0^E, (\Pi \Phi(f)^* \Pi)^n \Phi(f)^n \Omega_0^E \rangle_{\mathcal{E}} \\ &= \langle \Omega_0^E, \Phi(f)^{2n} \Omega_0^E \rangle_{\mathcal{E}} \\ &= (2n-1)!! \langle g, g \rangle^n_{\widetilde{\mathfrak{H}}_{-\frac{1}{2}}} \,. \end{aligned}$$

As in §III.1 of [21], we infer that the field $\widetilde{\varphi}$ can be expressed in terms of creation operators $\tilde{a}^*$ and (their adjoint) annihilation operators $\tilde{a}$, using the function $\nu(E)$ defined in (III.26). Specializing to the two-dimensional case, we have

$$\widetilde{\varphi}(t) = (2\pi)^{-1/2} \int \frac{dE}{\sqrt{2\nu(E)}} \big(\tilde{a}(E)^* + \tilde{a}(-E)\big)\, \mathrm{e}^{-iEt} \,.$$

IV. Classical Fields on the Cylinder $\boldsymbol{X} = \mathbb{R} \times \mathbb{T}^{d-1}$

In this section we study periodization of the spatial directions. We are especially interested in the spacetime $\boldsymbol{X} = \mathbb{R} \times \mathbb{T}^{d-1}$ with $\mathbb{T}^{d-1} = S^1 \times \cdots \times S^1$ denoting the $d-1$ dimensional torus. Let $\Lambda = \prod_{j=1}^{d-1} \ell_j$ denote the spatial volume, where ℓ_j is the circumference of the j^{th} constituent circle.

Define the quantum-mechanical Fock space

$$\mathcal{H}_\Lambda = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_n \ , \quad \text{where} \quad \mathcal{H}_n = \underbrace{\mathcal{H}_1 \otimes_s \cdots \otimes_s \mathcal{H}_1}_{n \text{ factors}} \ , \qquad \text{(IV.1)}$$

with $\mathcal{H}_1 = L_2(\Lambda; d\vec{x}\,)$. As in previous sections, the scalar $1 \in \mathbb{C}$ denotes the zero-particle vector Ω_0.

The positive Hamiltonians $H_\pm(\Lambda)$ arise as the quantization of the one-particle Hamiltonian $\mu_\pm$ given in (III.1). The form of the one-particle Hamiltonian $\mu_\pm(\vec{k})$ on the compactified spatial torus Λ is the same as that on Fourier space $\mathbb{R}^{d-1}$. For the model on the torus $\mathbb{T}^{d-1}$, however, the momenta $\vec{k}$ lie in the lattice $\mathcal{K}_\Lambda = \bigoplus_{j=1}^{d-1} \frac{2\pi}{\ell_j}\mathbb{Z}$ dual to $\mathbb{T}^{d-1}$.

The time-zero field

$$\varphi(\alpha) = \int \varphi(\vec{x})\alpha(\vec{x})\, d\vec{x} \ , \qquad \alpha \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{T}^{d-1}) \ .$$

acting on $\mathcal{H}_\Lambda$ has domain $\mathcal{D}(H_\pm(\Lambda))^{1/2} \subset \mathcal{D}(H_{\rm free}(\Lambda))^{1/2}$. Thus, the imaginary-time field

$$\varphi_I^\pm(t, \alpha) = \mathrm{e}^{-tH_\pm}\, \varphi(\alpha)\, \mathrm{e}^{tH_\pm} \ ,$$

has the dense domain $\mathrm{e}^{-(t+\epsilon)H_\pm}\mathcal{H}_\Lambda$ for any $\epsilon > 0$. As a form on $C^\infty(H_\pm) \times C^\infty(H_\pm)$,

$$\varphi_I^\pm(t, \vec{x}) = \frac{1}{\sqrt{\Lambda}} \sum_{k \in \mathcal{K}_\Lambda} \frac{1}{\sqrt{2\mu(\vec{k})}} \left(a(\vec{k})^* \mathrm{e}^{-t\mu_\mp(\vec{k})} + a(-\vec{k})\mathrm{e}^{t\mu_\pm(\vec{k})} \right) \mathrm{e}^{-i\vec{k}\cdot\vec{x}} \ .$$

Introducing the anti-time-ordering operator $\mathbb{A}$ for products of imaginary-time fields where, for example,

$$\mathbb{A}\varphi_I^\pm(x)\varphi_I^\pm(x') = \theta(t'-t)\varphi_I^\pm(x)\varphi_I^\pm(x') + \theta(t-t')\varphi_I^\pm(x')\varphi_I^\pm(x) \ ,$$

the two-point function $D_{\vec{v},\Lambda}$ can be expressed as

$$D_{\vec{v},\Lambda}(x - x') = \langle \Omega_0\, \mathbb{A}\varphi_I^\pm(x)\varphi_I^\pm(x')\Omega_0 \rangle_{\mathcal{H}_\Lambda} \ .$$

We present similar formulas for the completely compactified case in a later section.

Remark IV.1. The free Hamiltonian $H_\pm(\Lambda)$ on the spacetime $\mathbb{R} \times \Lambda$ is positive and has a trace-class heat kernel, which allows us to define the partition function

$$\begin{aligned}\mathfrak{Z}_{\pm,\beta,\Lambda} &= \mathrm{Tr}\ \mathrm{e}^{-\beta H_\pm(\Lambda)} \\ &= \prod_{\vec{k}\in\mathcal{K}_\Lambda} \frac{1}{1-\mathrm{e}^{-\beta\mu_\pm(\vec{k})}} \\ &= \prod_{\vec{k}\in\mathcal{K}_\Lambda} \Big(1+\rho_\pm(\vec{k})\Big)\ .\end{aligned}$$

The corresponding *Gibbs states* are

$$\langle\ \cdot\ \rangle_{\pm,\beta,\Lambda} = \frac{\mathrm{Tr}(\ \cdot\ \mathrm{e}^{-\beta H_\pm(\Lambda)})}{\mathfrak{Z}_{\pm,\beta,\Lambda}}\ .$$

Cyclicity of the trace shows that $\langle\ \cdot\ \rangle_{\beta,\pm,\Lambda}$ is invariant under the adjoint action of the unitary group $\mathrm{e}^{itH_\pm(\Lambda)}$. Furthermore, $\|\langle\ \cdot\ \rangle_{\beta,\pm,\Lambda}\| = 1$.

Positivity of the Hamiltonian $H_\pm$ implies that the functions

$$F^\pm_{A,B}(t) = \langle A\mathrm{e}^{itH_\pm}B\rangle_{\pm,\beta,\Lambda}\ , \qquad A,B\in\mathcal{B}(\mathcal{H}_\Lambda)\ ,$$

extend to analytic functions in the strip $z = t+is$ with $0<s<\beta$. Using the Hölder inequality for Schatten norms, one concludes that the function $F_\pm(z)$ is bounded uniformly inside the strip by $|F_\pm(z)| \leqslant \|A\|\,\|B\|$. Moreover, $F^\pm_{A,B}(t)$ satisfies the *KMS property*

$$F^\pm_{A,B}(t+i\beta) = \langle B\mathrm{e}^{-itH_\pm}A\rangle_{\pm,\beta,\Lambda}\ .$$

This property characterizes thermal states [24][25][14].

V. A Feynman-Kac Formula for $\mathscr{P}(\varphi)_2$ Models

In this section we establish a formula relating expectations of functions of complex fields to certain matrix elements of the heat kernel e^{-tH_v}. We call this a generalized Feynman-Kac formula, as the classical side of the identity is a sesquilinear form that is not densely defined. However, this classical form suffices to obtain a quantization which is the densely-defined heat kernel of the quantum-mechanical Hamiltonian. We use this generalized Feynman-Kac formula to establish useful bounds on the quantum-mechanical matrix elements.

In most usual cases one defines the classical side of the Feynman-Kac identity as matrix elements of a densely-defined form that determines an operator on a Hilbert space over path space. On our examples with interaction, the action functional $\mathfrak{A}(\Phi)$ is a normal operator on the space $\mathcal{E}$ of classical fields. However, we do not have complete information about the domain of $\Theta e^{-\mathfrak{A}}$, where Θ denotes the unitary time reflection on classical fields. We anticipate that this framework could be helpful in other contexts.

V.1. **Background.** Consider a spacetime cylinder with the time coordinate $t \in \mathbb{R}$ and spatial coordinate $x \in S^1$ parameterized by $x \in [-\frac{\ell}{2}, \frac{\ell}{2}]$ for $\ell > 0$. The $\mathscr{P}(\varphi)_2$-Hamiltonian is

$$H = H_{\text{free}} + H_{\text{int}} - E \;, \quad \text{where} \quad H_{\text{int}} = \int_{-\ell/2}^{\ell/2} dx \; :\mathscr{P}(\varphi(x)): \;, \tag{V.1}$$

where the polynomial $\mathscr{P}$ is bounded from below, and $: \cdot :$ denotes normal ordering with respect to the Fock space vacuum. The constant E equals the infimum of the spectrum of $H_{\text{free}} + H_{\text{int}}$, and $E \leqslant 0$. Define

$$H_v = H + Pv \;, \tag{V.2}$$

where $v = \tanh \beta$. Note that H_0 refers to the case $v = 0$, and not to the free Hamiltonian that we denote H_{free}.

Glimm and Jaffe showed in [9] that H with a spatial cutoff, rather than on a spatial circle, *i.e.*, with periodic boundary conditions, has a ground state Ω that is unique up to a phase. A similar proof applies to H with periodic boundary conditions. Since the Hamiltonian H and the momentum P commute, they can be simultaneously diagonalized. Both H_{free} and P have purely discrete spectra and compact resolvents. Following Heifets and Osipov, we have:

Proposition V.1. *The Hamiltonian H_v has pure discrete spectrum, and the heat kernel* e^{-tH_v} *is trace class on $\mathcal{H}$ for $t > 0$.*

Proof. Glimm and Jaffe [9] proved that given any $\epsilon > 0$, there exists $M_\epsilon < \infty$ such that $\epsilon H_{\text{free}} + H_{\text{int}} + M_\epsilon \geqslant 0$. Moreover,

$$|v| \, H_{\text{free}} \geqslant \pm Pv \;.$$

Choose $\epsilon > 0$, so $1 - 2\epsilon > |v|$. Hence, $(1-2\epsilon)H_{\text{free}} + Pv \geqslant 0$. It follows that

$$\begin{aligned} H_v &= ((1-2\epsilon)H_{\text{free}} + Pv) + (\epsilon H_{\text{free}} + H_{\text{int}} + M_\epsilon) + \epsilon H_{\text{free}} - E - M_\epsilon \\ &\geqslant \epsilon H_{\text{free}} - E - M_\epsilon \;. \end{aligned}$$

In other words, $\epsilon H_{\text{free}} \leqslant H_v + M_\epsilon + E$ and H_v is relatively compact with respect to H_{free}. Since H_0 has pure discrete spectrum, so does H_v. As the heat kernel $\mathrm{e}^{-tH_{\text{free}}}$ on the Hilbert space $\mathcal{H}$ is trace class for all $t > 0$, it follows that the heat kernel e^{-tH_v} is trace class on the Hilbert space $\mathcal{H}$ for all $t > 0$. □

V.2. **Operators, Forms, and the Feynman-Kac Formula.** In this section we investigate a Feynman-Kac formula for the matrix elements

$$\langle \widehat{\Omega}, \mathrm{e}^{-TH_v} \, \widehat{\Omega}' \rangle_{\mathcal{H}} = \langle \Omega, \Theta \mathrm{e}^{-\mathfrak{A}} \, \mathcal{T}(T) \Omega' \rangle_{\mathcal{E}} \;, \qquad T \in \mathbb{R}^+ \;, \tag{V.3}$$

of the heat kernel e^{-TH_v}, representing them as classical expectations of the time-reflection Θ times the exponential $e^{-\mathfrak{A}}$ of an action $\mathfrak{A}$ for a time interval of length T (corresponding to $t \in [-\frac{T}{2}, \frac{T}{2}]$), combined with the free unitary time-translation $t \mapsto \mathcal{T}(t)$ on $\mathcal{E}$. Up to an additive constant M, the action $\mathfrak{A}$ for the non-linear perturbation of the Hamiltonian is the integral[3] of a density $:\mathscr{P}(\Phi(t,\vec{x})):$, namely,

$$\mathfrak{A} = \int_{-T/2}^{T/2} dt \int_{-\ell/2}^{\ell/2} dx \, :\mathscr{P}(\Phi(t,x)): + \ell MT \, .$$

The value of the constant M is unimportant for our argument, except for the fact that it is finite, and it provides a normalizing factor that appears in all expectations. The function $\mathfrak{A}$, as well as its adjoint, are operators on $\mathcal{E}$ with the dense domain $\mathcal{D} \subset \mathcal{E}$ consisting of polynomials in $\Phi(f)$ applied to Ω_0^E, where $f \in C_0^\infty$. Write $\mathfrak{A} = \mathfrak{C} + i\mathfrak{D}$, where $\mathfrak{C}$ and $\mathfrak{D}$ are symmetric. Likewise, $\mathfrak{A}^* = \mathfrak{C} - i\mathfrak{D}$. Similarly, the operators

$$\begin{aligned}
\mathfrak{A}_+ &= \int_{0}^{T/2} dt \int_{-\ell/2}^{\ell/2} dx \, :\mathscr{P}(\Phi(t,x)): + \tfrac{1}{2}\ell MT \\
\mathfrak{A}_- &= \int_{-T/2}^{0} dt \int_{-\ell/2}^{\ell/2} dx \, :\mathscr{P}(\Phi(t,x)): + \tfrac{1}{2}\ell MT \\
\widetilde{\mathfrak{A}}_+ &= \int_{-T/2}^{T/2} dt \int_{0}^{\ell/2} dx \, :\mathscr{P}(\Phi(t,x)): + \tfrac{1}{2}\ell MT \\
\widetilde{\mathfrak{A}}_- &= \int_{-T/2}^{T/2} dt \int_{-\ell/2}^{0} dx \, :\mathscr{P}(\Phi(t,x)): + \tfrac{1}{2}\ell MT \, ,
\end{aligned}$$

as well as their real and imaginary parts $\mathfrak{C}_\pm, \mathfrak{D}_\pm, \widetilde{\mathfrak{C}}_\pm$, and $\widetilde{\mathfrak{D}}_\pm$, respectively, are all densely-defined, commuting operators on the domain $\mathcal{D}$.

Proposition V.2. *The operators $\mathfrak{A}$, $\mathfrak{A}_\pm$, and $\widetilde{\mathfrak{A}}_\pm$, as well as their adjoints, have normal, mutually commuting closures. We also denote them by $\mathfrak{A}$,* etc. *The polar decomposition of the exponential of $-\mathfrak{A}$ is $e^{-\mathfrak{A}} = e^{-\mathfrak{C}} e^{-i\mathfrak{D}}$. Similarly, $e^{-\mathfrak{A}_\pm} = e^{-\mathfrak{C}_\pm} e^{-i\mathfrak{D}_\pm}$ and $e^{-\widetilde{\mathfrak{A}}_\pm} = e^{-\widetilde{\mathfrak{C}}_\pm} e^{-i\widetilde{\mathfrak{D}}_\pm}$. Furthermore, $\Theta\mathfrak{A}_\pm = \mathfrak{A}_\mp^* \Theta$ and $\Theta e^{-\mathfrak{A}_\pm} = e^{-\mathfrak{A}_\mp^*}\Theta$, so*

$$\Theta e^{-\mathfrak{A}} = e^{-\mathfrak{A}_+^*} \Theta e^{-\mathfrak{A}_+} \, .$$

Proof. The fact that all these operators are defined on the domain $\mathcal{D}$ is a standard estimate relying on the fact that the covariance D_v is bounded from above by the covariance C; see §III.1. The proof of the

[3] In the case of a two-dimensional, cylindrical spacetime $\boldsymbol{X} = \mathbb{R} \times S^1$, we use the notation (t,x) for a point in time and space, dropping the vector on the spatial component $\vec{x}$.

normality and commutativity of the closures is similar to the proof of Proposition 2.1.2 in [8]. The Fock zero-particle state Ω_0^E is cyclic for the maximally-abelian von Neumann algebra generated by bounded functions of the fields $\Phi(f)$. This algebra commutes with the symmetric operators $\mathfrak{A}, \mathfrak{C}, \mathfrak{D}$, *etc.* It follows that $\mathfrak{C}$ and $\mathfrak{D}$ are essentially self-adjoint, their closures commute, and $\mathfrak{A}$ is normal. As a consequence of the definition of the fields, the relation $\Theta\mathfrak{A}_\pm = \mathfrak{A}_\mp^*\Theta$ is true on the domain $\mathcal{D}$. It extends to the closures by continuity, and hence to the exponentials defined by the functional calculus. Similar arguments apply to the other operators. □

While the vectors $A_{T/2}\Omega_0^E$, with $A_{T/2}(\Phi) = A(\Phi_{T/2})$ and $\Phi_{T/2}(t,x) = \Phi(t+\frac{T}{2}, x)$, may not be in the operator domain of $\mathrm{e}^{-\mathfrak{A}}$, we require that they be in the domain of $\Theta\mathrm{e}^{-\mathfrak{A}}$ or of $\Pi\mathrm{e}^{-\mathfrak{A}}$ as sesquilinear forms. The Feynman-Kac formula and the Schwarz inequalities we use involve such matrix elements. It is for this reason that we use the adjective *extended* for the Feynman-Kac formula.

Theorem V.3 (Feynman-Kac Density)**.** *The form domain of the operators* $\Theta\mathrm{e}^{-\mathfrak{A}}$ *and* $\Pi\mathrm{e}^{-\mathfrak{A}}$ *includes* $\mathcal{D}_0 \times \mathcal{D}_0 \subset \mathcal{E} \times \mathcal{E}$*, where* $\mathcal{D}_0 = \{\Phi(f_{T/2})^n\Omega_0^E\}$ *with* $f \in C_0^\infty$ *supported in the time-interval* $[0,1]\times[0,\frac{\ell}{2}]$*. For* $\Omega, \Omega' \in \mathcal{D}_0$*, the Feynman-Kac formula* (V.3) *holds. Moreover, one has the Schwarz inequalities*

$$\langle \Omega, \Pi\mathrm{e}^{-\mathfrak{A}}\Omega'\rangle_{\mathcal{E}} \leqslant \langle \Omega, \Pi\mathrm{e}^{-\mathfrak{A}}\Omega\rangle_{\mathcal{E}}^{1/2}\langle \Omega', \Pi\mathrm{e}^{-\mathfrak{A}}\Omega'\rangle_{\mathcal{E}}^{1/2}\ , \tag{V.4}$$

and

$$\langle \Omega, \Theta\mathrm{e}^{-\mathfrak{A}}\Omega'\rangle_{\mathcal{E}} \leqslant \langle \Omega, \Theta\mathrm{e}^{-\mathfrak{A}}\Omega\rangle_{\mathcal{E}}^{1/2}\langle \Omega', \Theta\mathrm{e}^{-\mathfrak{A}}\Omega'\rangle_{\mathcal{E}}^{1/2}\ . \tag{V.5}$$

Lemma V.4. *Let* $A, B, A+B$ *be essentially self-adjoint operators on a Hilbert space* $\mathcal{H}$*, each bounded from above. Let* C *be self-adjoint and commuting with the closures of* A, B*, and* $A+B$*. Then*

$$\|C\mathrm{e}^{A+B}\| \leqslant \|\mathrm{e}^{A}\|\ \|C\mathrm{e}^{B}\|\ . \tag{V.6}$$

Proof. First, consider the case that $C = \mathbb{1}$. As A and B are bounded from above, the operators e^A, e^B, and e^{A+B} are each bounded, too. Moreover, essential self-adjointness assures that Chernoff's version of the Trotter product formula applies; see the Corollary to the Theorem in [6]. Therefore,

$$\mathrm{e}^{A+B} = \mathrm{st.}\lim_{n\to\infty}\left(\mathrm{e}^{A/n}\,\mathrm{e}^{B/n}\right)^n\ .$$

Consequently,

$$\begin{aligned}\|\mathrm{e}^{A+B}\| &= \Big\|\mathrm{st.}\lim_{n\to\infty}\big(\mathrm{e}^{A/n}\,\mathrm{e}^{B/n}\big)^n\Big\| \\ &\leqslant \lim_{n\to\infty}\|\mathrm{e}^{A/n}\|^n\;\|\mathrm{e}^{B/n}\|^n\;.\end{aligned}$$

If M_A denotes the upper bound of the spectrum of A, then the spectral theorem implies $\|\mathrm{e}^{A/n}\| = \mathrm{e}^{M_A/n}$, and similarly for B. The bound (V.6) for $C = \mathbb{1}$ now follows.

Consider the case that $C \neq \mathbb{1}$. Unless $C\mathrm{e}^B$ is bounded, there is nothing to prove. As $\|T\| = \|TT^*\|^{1/2}$, one infers

$$\|C\mathrm{e}^{A+B}\| = \|C\mathrm{e}^{2A+2B}C\|^{1/2} = \|C^2\mathrm{e}^{2A+2B}\|^{1/2}\;.$$

As in the proof with $C = \mathbb{1}$, one has on the domain $\mathcal{D}\times\mathcal{D}$ with $\mathcal{D}$ equal to the domain of C^2, and

$$C^2\,\mathrm{e}^{2A+2B} = C^2\,\mathrm{st.}\lim_{n\to\infty}\big(\mathrm{e}^{2A/n}\,\mathrm{e}^{2B/n}\big)^n = \mathrm{w.}\lim_{n\to\infty}\big(\mathrm{e}^{2A/n}\,C^{2/n}\mathrm{e}^{2B/n}\big)^n\;.$$

Thus,

$$\|C\mathrm{e}^{A+B}\| \leqslant \lim_{n\to\infty}\|\mathrm{e}^{2A/n}\|^{n/2}\;\|C^{2/n}\mathrm{e}^{2B/n}\|^{n/2}\;.$$

As above, $\|\mathrm{e}^{2A/n}\| = \mathrm{e}^{2M_A/n} = \|\mathrm{e}^A\|^{2/n}$. As B and C^2 commute, they can be simultaneously diagonalized, so $\|C^{2/n}\mathrm{e}^{2B/n}\| = \|C\mathrm{e}^B\|^{2/n}$. Hence, the general case of (V.6) also holds. □

Let H_v be given by (V.2).

Lemma V.5. *The heat kernel* e^{-tH_v} *is the boundary value of an analytic function of* v *in the disk* $|v| < 1$.

Proof. Choose a maximum value of $|v| = \Gamma < 1$, and choose $0 < \epsilon < 1-\Gamma$. Note that

$$\left.\frac{d^n}{dv^n}\,\mathrm{e}^{-tH_v}\right|_{v=0} = (-tP)^n\mathrm{e}^{-tH_0}.$$

Let $A = -t(\epsilon H_{\mathrm{free}} + H_I - E)$, $B = -t(1-\epsilon)H_{\mathrm{free}}$, and $C = (-tvP)^n$. The operators A, B, and $A+B = -tH$ are essentially-self adjoint on $C^\infty(H_{\mathrm{free}})$; see [26]. For given t,Γ they are bounded from above, and also P commutes with A, B and their closures. Moreover,

$$\begin{aligned}\frac{|v|^n}{n!}\left\|\left.\frac{d^n}{dv'^n}\,\mathrm{e}^{-tH_{v'}}\right|_{v'=0}\right\| &= \frac{1}{n!}\,\big\|(tvP)^n\mathrm{e}^{-tH_0}\big\| \\ &= \frac{1}{n!}\,\big\|C\mathrm{e}^{(A+B)}\big\|\;.\end{aligned}$$

Thus, we can apply Lemma V.4 to obtain

$$\frac{|v|^n}{n!} \left\| \frac{d^n}{dv'^n} \mathrm{e}^{-tH_{v'}} \Big|_{v'=0} \right\| \leqslant \frac{1}{n!} \, \|\mathrm{e}^A\| \, \|C\mathrm{e}^B\| \ .$$

Since $|v| \leqslant \Gamma$ and $|P|^n \leqslant H_{\mathrm{free}}^n$, we infer

$$\begin{aligned} \|C\,\mathrm{e}^B\| &= \|(tvP)^n \, \mathrm{e}^{-t(1-\epsilon)H_{\mathrm{free}}}\| \\ &\leqslant \left(\frac{|v|}{1-\epsilon}\right)^n n! \\ &\leqslant \left(\frac{\Gamma}{1-\epsilon}\right)^n n! \ . \end{aligned}$$

Combining these bounds,

$$\begin{aligned} \frac{|v|^n}{n!} \left\| \frac{d^n}{dv'^n} \mathrm{e}^{-tH_{v'}} \Big|_{v'=0} \right\| &\leqslant \frac{1}{n!} \, \|\mathrm{e}^A\| \, \|C\,\mathrm{e}^B\| \\ &\leqslant \left(\frac{\Gamma}{1-\epsilon}\right)^n \|\mathrm{e}^A\| \ . \end{aligned}$$

As $\Gamma < 1-\epsilon$, and e^A is bounded, the Taylor series

$$\mathrm{e}^{-tH_v} = \sum_{n=0}^{\infty} \frac{v^n}{n!} \left(\frac{d^n}{dv'^n}\right) \mathrm{e}^{-tH_{v'}} \Big|_{v'=0} \tag{V.7}$$

is norm-convergent in the disk $|v| \leqslant \Gamma$. But $\Gamma < 1$ is arbitrary, so the heat kernel is analytic in v throughout the open unit disk. □

Proof of Theorem V.3. We only consider $\Theta \mathrm{e}^{-\mathfrak{A}}$, as the argument for $\Pi \mathrm{e}^{-\mathfrak{A}}$ is similar. In the case that $v = 0$, the proposition holds as a consequence of the standard Feynman-Kac formula. The bound is just the Schwarz inequality in the reflection-positive inner product defined by one or the other Osterwalder-Schrader quantization. From Lemma V.5, we infer that

$$v \mapsto \langle \Omega, \Theta \mathrm{e}^{-\mathfrak{A}} \Omega' \rangle_{\mathcal{E}} = \left\langle \widehat{\Omega}, \mathrm{e}^{-TH_v} \widehat{\Omega}' \right\rangle_{\mathcal{H}}$$

is the boundary value of an analytic function of v in the unit disk. We can identify this function for v purely imaginary and small in magnitude as a matrix element expectation of the heat kernel e^{-TH_0} in the vector $\widehat{\Omega}$ and a translate of the vector $\widehat{\Omega}'$. Once the expectation is defined for real v, the Schwarz inequality follows from the positivity of the reflection-positive inner product. □

VI. The Spectrum Condition on the Cylinder

In this section we consider quantum fields on the two-dimensional, cylindrical spacetime $\mathbb{R} \times S^1$. We take $t \in \mathbb{R}$ as the time coordinate, parameterized by the real line, and $x \in [-\frac{\ell}{2}, \frac{\ell}{2}]$ as the spatial coordinate, parametrized by the circle. We consider the $\mathscr{P}(\varphi)_2$-Hamiltonian H given in (V.1) and the Hamiltonian $H_v = H + Pv$ defined in (V.2).

Fourier analysis shows that the spectrum condition $0 \leqslant H_v$ holds for free fields. Glimm and Jaffe conjectured that the spectrum condition also holds for Hamiltonians defined on the spatial circle [10]. Heifets and Osipov later proved this both for $\mathscr{P}(\varphi)_2$ models [15] and for the Yukawa$_2$ model [16]. While Glimm and Jaffe believed to have a proof of the spectrum condition in [10], they had not established uniqueness of the ground state Ω_v for H_v; see the footnote on p. 1584 of [11]. Our construction with complex classical fields [21] provides a context for the Heifets-Osipov argument showing that Ω_v is unique.

A consequence of this spectrum condition for theories on a spatial circle is analyticity of the Schwinger functions in the spatial directions for $|x_i - x_{i+1}| < |t_i - t_{i+1}|$.

Proposition VI.1 (Spectrum Condition on a Cylinder). *For real v, with $|v| < 1$, one has $0 \leqslant H_v$. Any ground state Ω_v of H_v is a multiple of Ω, and $H_v\Omega = P\Omega = 0$.*

Proof. The proof of the spectrum condition $0 \leqslant H_v$ has two distinct parts. The first part is to reduce the proof to the bound

$$\langle \Psi, \mathrm{e}^{-TH_v}\Psi\rangle_{\mathcal{H}} \quad\leqslant\quad \langle \Omega_0, \mathrm{e}^{-TH_v}\Omega_0\rangle_{\mathcal{H}}^{1/2}\ \langle \Psi', \mathrm{e}^{-TH_v}\Psi'\rangle_{\mathcal{H}}^{1/2} \tag{VI.1}$$

for a dense set of vectors $\{\Psi\}$, with $\Psi' = \Psi'(\Psi)$, and with Ω_0 denoting the Fock zero-particle vector. The second part is to prove (VI.1).

First Part: We give a variation of the argument in Heifets and Osipov [15]. As recalled in the beginning of §V, the Hamiltonian H has a unique ground state Ω, up to a phase, and H_v has a negative ground-state energy E_v. Assume (VI.1). Then

$$\begin{aligned}\langle \Psi, \mathrm{e}^{-T(H_v-E_v)}\Psi\rangle_{\mathcal{H}} &\leqslant \langle \Omega_0, \mathrm{e}^{-T(H-E_v)}\Omega_0\rangle_{\mathcal{H}}^{1/2}\ \langle \Psi', \mathrm{e}^{-T(H_v-E_v)}\Psi'\rangle_{\mathcal{H}}^{1/2}\\ &\leqslant \mathrm{e}^{TE_v/2}\, \|\Psi'\|\ ,\end{aligned} \tag{VI.2}$$

where we have used $P\Omega_0 = 0$ to bound the first expectation and $0 \leqslant H_v - E_v$ to bound the second one. Thus, for a dense set of vectors $\{\Psi\}$, the expectation of $\mathrm{e}^{-T(H_v-E_v)}$ decays with the exponential rate $\mathrm{e}^{-T|E_v|/2}$ as $T \to \infty$. Hence, $\frac{1}{2}|E_v| \leqslant H_v - E_v$. However, $H_v - E_v$ has a zero energy ground state, so it follows that $E_v = 0$.

Now consider whether the ground state vector is unique. Denote one ground state vector of H_v by Ω, and let Ω' be a second ground state. For each value of v and for $0 < \epsilon < 1 - \tanh|v|$, one has $v_\pm$ for which $\tanh v_\pm = \tanh v \pm \epsilon$. Hence, $0 \leqslant H_{v_+} H_{v_-}$, that is,

$$\epsilon^2 P^2 \leqslant H_v^2 \ . \tag{VI.3}$$

Taking the expectation of (VI.3) in the vector Ω' shows $P\Omega' = 0$. It follows that Ω' is also a ground state of H, whose ground state is unique. Thus, $\Omega' = \alpha\Omega$, where α is a phase.

Second Part: Let $A(\varphi)$ denote a polynomial function of the imaginary-time field, with each field averaged with a C_0^∞-function supported in the rectangle $(t, x) \in [0, \epsilon] \times [0, \frac{\ell}{2}]$. Then $\Psi = A(\varphi)\Omega_0 = (A(\Phi)\Omega_0^E)^\wedge$ is the quantization from temporal-reflection positivity. Note that Theorem II.3 ensures that the states Ψ are dense in $\mathcal{H}$. The vector $\mathrm{e}^{-TH_v/2}\Psi$ is the quantization of $\mathrm{e}^{-\mathfrak{A}_+} A_{T/2}\Omega_0^E$. The Feynman-Kac formula based on temporal-reflection positivity, elaborated in §V.2, is

$$\langle \Psi, \mathrm{e}^{-TH_v}\Psi\rangle_{\mathcal{H}} = \langle A_{T/2}\Omega_0^E, \Theta \mathrm{e}^{-\mathfrak{A}}\, A_{T/2}\Omega_0^E\rangle_{\mathcal{E}} \ . \tag{VI.4}$$

The action $\mathfrak{A} = \mathfrak{A}_+ + \mathfrak{A}_-$ is the sum of positive-time and negative-time parts $\mathfrak{A}_\pm$, with $\mathfrak{A}_+^{*\Theta} = \Theta\mathfrak{A}_+^*\Theta = \mathfrak{A}_-$ and $\mathfrak{A}_+^{*\Pi} = \Pi\mathfrak{A}_+^*\Pi = \mathfrak{A}_-$. It is also a sum of parts $\widetilde{\mathfrak{A}}_\pm$ localized at positive or negative spatial coordinate x, namely, $\mathfrak{A} = \widetilde{\mathfrak{A}}_+ + \widetilde{\mathfrak{A}}_-$, with $\Pi\widetilde{\mathfrak{A}}_+^*\Pi = \widetilde{\mathfrak{A}}_-$. Let $A'' = A_{T/2}^{*\Theta} A_{T/2}$, $A' = A^{*\Pi}A$, and $\Psi' = (A'\Omega_0^E)^\wedge$, where $A_{T/2}^{*\Theta} = \Theta A_{T/2}^*\Theta$ and $A^{*\Pi} = \Pi A^*\Pi$. Spatial-reflection positivity, along with invariance of Ω_0^E under Θ and Π and time-translation, implies

$$\begin{aligned}
\langle \Psi, \mathrm{e}^{-TH_v}\Psi\rangle_{\mathcal{H}} &= \langle \Omega_0^E, \Theta\mathrm{e}^{-\mathfrak{A}}\, A''\Omega_0^E\rangle_{\mathcal{E}} = \langle \Omega_0^E, \Pi\mathrm{e}^{-\mathfrak{A}}\, A''\Omega_0^E\rangle_{\mathcal{E}} \\
&\leqslant \langle \Omega_0^E, \Pi\mathrm{e}^{-\mathfrak{A}}\, \Omega_0^E\rangle_{\mathcal{E}}^{1/2} \langle A''\Omega_0^E, \Pi\mathrm{e}^{-\mathfrak{A}}\, A''\Omega_0^E\rangle_{\mathcal{E}}^{1/2} \\
&= \langle \Omega_0^E, \Theta\mathrm{e}^{-\mathfrak{A}}\, \Omega_0^E\rangle_{\mathcal{E}}^{1/2} \langle A'_{T/2}\Omega_0^E, \Theta\mathrm{e}^{-\mathfrak{A}}\, A'_{T/2}\Omega_0^E\rangle_{\mathcal{E}}^{1/2} \\
&= \langle \Omega_0, \mathrm{e}^{-TH_v}\Omega_0\rangle_{\mathcal{H}}^{1/2} \langle \Psi', \mathrm{e}^{-TH_v}\Psi'\rangle_{\mathcal{H}}^{1/2} \ .
\end{aligned} \tag{VI.5}$$

Note that we use the identity

$$A''^{*\Pi}A'' = A'^{*\Theta}_{T/2}A'_{T/2} = A^{\Theta\Pi}_{T/2}A^{*\Theta}_{T/2}A^{*\Pi}_{T/2}A_{T/2}.$$

In the Feynman-Kac formula and the inequality (VI.5), we have used Proposition V.3. In (V.4), we take $\Omega_1 = \Omega_0^E$ and $\Omega_2 = A''\Omega_0^E$. □

VII. Classical Fields on the Cylinder $\boldsymbol{X} = S^1 \times \mathbb{R}^{d-1}$

In this section we study periodization of time for the classical field. We are especially interested in the spacetime $\boldsymbol{X} = S^1 \times \mathbb{R}^{d-1}$ with S^1 parameterized[4] by $t \in [-\frac{\beta}{2}, \frac{\beta}{2}]$. As investigated by Høegh-Krohn [17], these classical fields yield a positive temperature state of the quantum field. Define the positive-time and negative-time half-circles $S^1_+ = [0, \frac{\beta}{2}]$ and $S^1_- = [-\frac{\beta}{2}, 0]$, and the positive and negative time half-spaces

$$\boldsymbol{X}_\pm = S^1_\pm \times \mathbb{R}^{d-1} \,. \tag{VII.1}$$

Their intersection

$$\boldsymbol{X}_+ \cap \boldsymbol{X}_- = (\{0\} \times \mathbb{R}^{d-1}) \cup (\{\tfrac{\beta}{2}\} \times \mathbb{R}^{d-1}) \tag{VII.2}$$

consists of two disjoint copies of $\mathbb{R}^{d-1}$ forming the boundary $\partial \boldsymbol{X}_+$ of $\boldsymbol{X}_+$.

VII.1. **Reflection Positivity.** As proved in Proposition VI.3 of [21], reflection positivity of $D_{\vec{v}}$ carries over to reflection positivity of

$$D^c(x, x') = \sum_{n=-\infty}^{\infty} D_{\vec{v}}(t - t' + n\beta, \vec{x} - \vec{x}\,') \,. \tag{VII.3}$$

Thus, temporal-reflection positivity, $0 \leqslant \vartheta D_{\vec{v}}$ on $L_{2,+}$, established in Proposition III.3, ensures

$$0 \leqslant \vartheta D^c \quad \text{on} \quad \mathcal{K}_\pm = L_2(\boldsymbol{X}_\pm) \,.$$

Therefore, one can quantize functions supported in the positive and negative-time half-spaces $\boldsymbol{X}_\pm$, using the reflection positive kernels ϑD^c or $D^c \vartheta$ on $\mathcal{K}_\pm$ for the neutral scalar field.

For the two-point function $D_{\vec{v}}$ on $L_2(\mathbb{R}^d)$ defined in (III.5) periodization yields, using (III.6),

$$\begin{aligned} D^c(x, x') &= \sum_{n \in \mathbb{Z}} \frac{\mathrm{e}^{-\mu\,|t-t'+n\beta| + \delta(t-t'+n\beta)}}{2\mu} (\vec{x} - \vec{x}\,') \qquad \text{(VII.4)} \\ &= \begin{cases} \frac{1}{2\mu} \left(\frac{\mathrm{e}^{-(t-t')\mu_-}}{1-\mathrm{e}^{-\beta\mu_-}} + \frac{\mathrm{e}^{-(\beta-(t-t'))\mu_+}}{1-\mathrm{e}^{-\beta\mu_+}} \right) (\vec{x} - \vec{x}\,') \,, & \text{if } t > t', \\ \frac{1}{2\mu} \left(\frac{\mathrm{e}^{(t-t')\mu_+}}{1-\mathrm{e}^{-\beta\mu_+}} + \frac{\mathrm{e}^{-(\beta+(t-t'))\mu_-}}{1-\mathrm{e}^{-\beta\mu_-}} \right) (\vec{x} - \vec{x}\,') \,, & \text{if } t < t', \end{cases} \end{aligned}$$

where $\mu_\pm = \mu \pm \delta$ is defined in (III.1).

[4] Here and in other sections describing positive-temperature fields, we denote $\beta = \frac{1}{k_{\mathrm{B}} T}$ as the inverse temperature. This is in contrast with the notation in §VI, where $\tanh |\beta| = |\vec{v}|$ denotes the velocity of a Lorentz boost.

Interpretation. In two different ways, one can connect a point with negative time, $-s \in [-\frac{\beta}{2}, 0]$, to a point with positive time, $s' \in [0, \frac{\beta}{2}]$. A trajectory of minimal length can pass through time $t = 0$, or it can pass through time $t = \frac{\beta}{2}$. The minimal length of a trajectory through $t = 0$ is $s + s'$, while that through $t = \frac{\beta}{2}$ is $\beta - s - s'$. A trajectory may also wind $n \geqslant 1$ times around the circle, adding a length of $n\beta$ to the minimum. Each of these possibilities contributes a term to the Green's function with an exponential decay rate equal to the minimal length times a corresponding Hamiltonian. In fact, the full Green's function is a sum of such terms. In particular, the minimal trajectory through $t = 0$ contributes the heat kernel $\mathrm{e}^{-(s+s')\mu_+}$ to the Green's function. As the length of the trajectory $s + s'$ increases with the increase of either s or s', the corresponding decay rate increases as well; therefore the Hamiltonian μ_+ is positive. The factor

$$\sum_{n=0}^{\infty} \mathrm{e}^{-n\beta\mu_+} = (1 - \mathrm{e}^{-\beta\mu_+})^{-1}$$

in the inner product reflects a correction from including terms labelled by trajectories that make $n \geqslant 0$ multiple circuits around the circle. This factor affects the normalization of the state rather than the Hamiltonian.

The second term in the Green's function arises from terms labelled by the trajectories through $t = \frac{\beta}{2}$. The corresponding minimal trajectory gives the heat kernel $\mathrm{e}^{-(\beta-s-s')\mu_-}$. The separation $\beta - s - s'$ decreases with increasing $s, s' \in [0, \frac{\beta}{2}]$, so the corresponding Hamiltonian $-\mu_-$ is negative. The correction factor $1 + \rho_- = (1 - \mathrm{e}^{-\beta\mu_-})^{-1}$ arises from the sum over circuits from s' to $-s$ that travel the minimal distance, but with the constraint that they pass through time $t = \frac{\beta}{2}$, followed by multiple complete circuits around the circle.

The operators $\mu_\pm = \mu \pm \delta$ acting on the Hilbert space $L_2(\mathbb{R}^{d-1})$ are self-adjoint and positive. They satisfy the lower bound $m\sqrt{1 - \vec{v}^{\,2}} \leqslant \mu_\pm$ of (III.3). Hence,

$$\|\mathrm{e}^{-\beta\mu_\pm}\| \leqslant \mathrm{e}^{-m\beta\sqrt{1-\vec{v}^{\,2}}}$$

and

$$\| \left(\mathbb{1} - \mathrm{e}^{-\beta\mu_\pm}\right)^{-1} \| \leqslant \left(1 - \mathrm{e}^{-m\beta\sqrt{1-\vec{v}^{\,2}}}\right)^{-1} .$$

Consequently, for fixed t and t', the sum over $n \in \mathbb{Z}$ in (VII.4) is norm-convergent in the sense of operators on $L_2(\mathbb{R}^{d-1})$. It is common to define

$$\rho_\pm = \frac{\mathrm{e}^{-\beta\mu_\pm}}{\mathbb{1} - \mathrm{e}^{-\beta\mu_\pm}} \;, \qquad \text{so} \quad \mathbb{1} + \rho_\pm = \frac{1}{\mathbb{1} - \mathrm{e}^{-\beta\mu_\pm}} \;.$$

Note that

$$\|\rho_\pm\| \leqslant (\mathrm{e}^{m\beta\sqrt{1-\vec{v}^{\,2}}} - 1)^{-1} \;. \tag{VII.5}$$

This allows us to rewrite (VII.4), for $x, x' \in \boldsymbol{X}_+$, as

$$(\vartheta D^c)(x,x') = \frac{1}{2\mu}\Big((\mathbb{1}+\rho_+)\mathrm{e}^{-(t+t')\mu_+} + \rho_-\mathrm{e}^{(t+t')\mu_-}\Big)(\vec{x}-\vec{x}\,') \;. \tag{VII.6}$$

The operators $\rho_\pm$ commute with μ, so their norms on $\mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$ agree with their norms on $L_2(\mathbb{R}^{d-1})$. In Fourier space the function $\mu(\vec{k})$ is real and even, so the operator μ in configuration space is real, acting on either $L_2(\mathbb{R}^{d-1})$ or on $\mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$. Similarly, $\delta(k)$ is real and odd, so the operator δ in configuration space is purely imaginary, *viz.*, $\overline{\delta} = -\delta$. Consequently, $\overline{\mu_\pm} = \mu_\mp$ and, setting $f_t(\vec{x}) = f(t,\vec{x})$, one has

$$\begin{aligned}\big\langle f, \vartheta D^c g\big\rangle_{\mathcal{K}} &= \Big\langle \int_0^{\beta/2} \mathrm{e}^{-t\mu_+} f_t\,dt \;, (\mathbb{1}+\rho_+)\int_0^{\beta/2} \mathrm{e}^{-t'\mu_+} g_{t'}\,dt'\Big\rangle_{\mathfrak{H}_{-\frac12}} \\ &\quad + \Big\langle \int_0^{\beta/2} \overline{\mathrm{e}^{t\mu_+}\rho_+^{1/2}} f_t\,dt \;, \int_0^{\beta/2} \overline{\mathrm{e}^{t\mu_+}\rho_+^{1/2}}\, g_{t'}\,dt'\Big\rangle_{\mathfrak{H}_{-\frac12}} \;. \end{aligned} \tag{VII.7}$$

Remark VII.1. Let $s \in \mathbb{R}$, $0 \leqslant t \leqslant \frac{\beta}{2}$ and $\alpha \in \mathfrak{H}_{-\frac12}$. Eq. (VII.6) suggests to consider the analytic map

$$s + it \mapsto \Big(\mathrm{e}^{i(s+it)\mu_+}(\mathbb{1}+\rho_+)^{1/2}\alpha \;,\; \mathrm{e}^{-i(s+it)\mu_-}\rho_-^{1/2}\alpha\Big) \in \mathfrak{H}_{-\frac12} \oplus \mathfrak{H}_{-\frac12} \;.$$

Note that the action of the time-evolution $s \mapsto \mathrm{e}^{is\mu_+}$ in the second component is oriented toward the past, and μ_+ is replaced by its complex conjugate $\mu_- = \overline{\mu_+}$. Both aspects can be avoided by using a complex conjugate Hilbert space in the second component, as suggested by the equivalent formula for $\big\langle f, \vartheta D^c g\big\rangle_{\mathcal{K}}$ given in Eq. (VII.12); see Section VII.3.

VII.2. **Estimates on the Kernels.** The operator D^c can be decomposed into its hermitian and skew-hermitian parts on $\mathcal{K}$,

$$D^c = K^c + iL^c \;.$$

The operators D^c, K^c, L^c and their kernels $D^c(x,x')$, $K^c(x,x')$, *etc.*, have elementary Fourier representations, almost identical to the expression for the continuum Fourier transform, such as (VII.8) in the case of $D_{\vec{v}}(x,x')$. As a consequence, this entails operator bounds similar to those established in §III.1 prior to compactification. We formulate these properties in the following proposition. Let us denote the lattice of energy values dual to $[0,\beta]$ by $\mathcal{L}_\beta = \frac{2\pi}{\beta}\mathbb{Z}$, known as *Matsubara frequencies* in the physics literature.

Theorem VII.2. *The kernel $D^c(x,x')$ has the representation*

$$D^c(x,x') = \frac{1}{\beta}\sum_{E\in\mathcal{L}_\beta}\frac{1}{(2\pi)^{d-1}}\int_{\mathbb{R}^{d-1}} d\vec{k}\,\frac{\mathrm{e}^{iE(t-t')+i\vec{k}(\vec{x}-\vec{x}')}}{(E+i\delta)^2+\mu(\vec{k})^2}\ . \tag{VII.8}$$

The operators K^c, L^c, D^c, $|D^c|$, and C^c mutually commute and satisfy

$$K^c \leqslant |D^c| \leqslant (1-\vec{v}^{\,2})^{-1/2}\,K^c\ .$$

In particular, D^c is invertible. Moreover,

$$\tfrac{1}{2}\left(1-\vec{v}^{\,2}\right)C^c < K^c < (1-\vec{v}^{\,2})^{-2}\,C^c \tag{VII.9}$$

and $\sup_k\left|\frac{\widetilde{L}^c(k)}{\widetilde{K}^c(k)}\right| = \frac{|\vec{v}|}{\sqrt{1-\vec{v}^{\,2}}}$. *Also,*

$$\tfrac{1}{2}\left(1-\vec{v}^{\,2}\right)C^c < |D^c| < (1-\vec{v}^{\,2})^{-5/2}\,C^c\ . \tag{VII.10}$$

Proof. The operator $D^c(x,x')$ is periodic in $t-t'$ with period β. To establish its Fourier representation, we consider the Fourier series in the variable $\xi = t-t'$. Similar to the proof of Proposition II.4 of [19], but for operators,

$$(\mathbb{1}+\rho_-)\int_0^\beta \mathrm{e}^{-(\mu_-+iE)\xi}\,d\xi = \frac{1}{\mu_-+iE}\ ,$$
$$\int_0^\beta \mathrm{e}^{(\mu_+-iE)\xi}\rho_+\,d\xi = \frac{1}{\mu_+-iE}\ ,$$

and $\mu_+ + \mu_- = 2\mu$. Use the representation (VII.4) for $\xi \geqslant 0$. One has the E-dependent operator D_E on $\mathfrak{H}_{-\frac{1}{2}}$ defined by

$$D_E(x,x') = \int_0^\beta D^c(x,x')\mathrm{e}^{-iE\xi}\,d\xi\,.$$

It follows that

$$\begin{aligned} D_E &= \frac{1}{2\mu}\left(\frac{1}{\mu_-+iE}+\frac{1}{\mu_+-iE}\right)\\ &= \frac{1}{(\mu_-+iE)(\mu_+-iE)}\\ &= \frac{1}{(E+i\delta)^2+\mu^2}\ . \end{aligned}$$

In terms of the integral kernel $D_E(\vec{x}-\vec{x}')$, one has

$$D^c(x,x') = \frac{1}{\beta}\sum_{E\in\mathcal{L}_\beta} \mathrm{e}^{iE(t-t')}\, D_E(\vec{x}-\vec{x}')\ ,$$

which is equivalent to (VII.8).

The operators K^c, L^c, D^c, and C^c are all translation invariant and hence mutually commute, as well as with their adjoints and functions defined by the spectral theorem. The remaining statements are estimates on the operators that follow by comparing their Fourier representations. Since the Fourier representation of $D^c(x,x')$ agrees with that of $D_{\vec{v}}(x,x')$ for specific values of E, the inequalities established in §III.1 hold here as well, after translation of the constants to the present notation by substituting $\tanh|\beta| \mapsto |\vec{v}|$. □

Corollary VII.3. *The operator D^c extends to $\mathfrak{H}_{-1}(\boldsymbol{X})$.*

Proof. Proposition VII.2 ensures that D^c has a square root $D^{c\,\frac{1}{2}}$. Denote the constant $(1-\vec{v}^{\,2})^{-5/2}$ in (VII.10) of Proposition VII.2 by M^2. It follows that

$$\begin{aligned}\|D^{c\,\frac{1}{2}} f\|_{\mathcal{K}} &= \langle f, |D^c| f\rangle_{\mathcal{K}}^{1/2} \\ &\leqslant M\langle f, C^c f\rangle_{\mathcal{K}}^{1/2} \\ &= M\|f\|_{\mathfrak{H}_{-1}(\boldsymbol{X}^c)}\ .\end{aligned} \tag{VII.11}$$

Hence, D^c extends to $\mathfrak{H}_{-1}(\boldsymbol{X})$. □

Note that the function $\delta_s\otimes\alpha$ has Fourier representation proportional to $\mathrm{e}^{-isE}\,\widetilde{\alpha}(\vec{k})$. Thus,

$$\|\delta_s\otimes\alpha\|^2_{\mathfrak{H}_{-1}(\boldsymbol{X}^c)} = \frac{1}{\beta}\sum_{E\in\mathcal{L}_\beta}\int_{\mathbb{R}^{d-1}} \frac{|\widetilde{\alpha}(\vec{k})|^2}{E^2+\mu(\vec{k})^2}\, d\vec{k}\ .$$

This sum is real and positive, so there is a constant $\widetilde{M} = 1 + O(\frac{1}{\beta})$ such that it can be estimated by a multiple of the Riemann integral it approximates. Then

$$\begin{aligned}\|\delta_s\otimes\alpha\|^2_{\mathfrak{H}_{-1}(\boldsymbol{X}^c)} &\leqslant \frac{\widetilde{M}}{2\pi}\int_{E\in\mathbb{R}} dE \int_{\mathbb{R}^{d-1}} \frac{|\widetilde{\alpha}(\vec{k})|^2}{E^2+\mu(\vec{k})^2}\, d\vec{k} \\ &= \widetilde{M}\|\alpha\|^2_{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})}\ .\end{aligned}$$

Thus, for $\alpha\in\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$, the generalized function $\delta_s\otimes\alpha$ is an element of the Sobolev space $\mathfrak{H}_{-1}(\boldsymbol{X})$.

VII.3. **Thermal Quantization Maps.** We can write (VII.7) as

$$\langle f, \vartheta D^c g \rangle_{\mathcal{K}} = \Big\langle \int_0^{\beta/2} \mathrm{e}^{-t\mu_+} f_t \, dt \,, (\mathbb{1} + \rho_+) \int_0^{\beta/2} \mathrm{e}^{-t'\mu_+} g_{t'} \, dt' \Big\rangle_{\mathfrak{H}_{-\frac{1}{2}}} + \Big\langle \int_0^{\beta/2} \mathrm{e}^{t\mu_+} \rho_+^{\frac{1}{2}} \, \overline{g_t} \, dt \,, \int_0^{\beta/2} \mathrm{e}^{t\mu_+} \rho_+^{\frac{1}{2}} \, \overline{f_{t'}} \, dt' \Big\rangle_{\mathfrak{H}_{-\frac{1}{2}}} \,, \quad \text{(VII.12)}$$

where $\overline{f}$ denotes the complex conjugate of the function f. The two terms appearing on the right hand side in (VII.12) are related to the two disjoint components of the boundary $\partial \boldsymbol{X}_+$ of $\boldsymbol{X}_+$ as discussed in the interpretation provided in Section VII.1. The special form of these two terms can be accommodated for by the following points:

i.) Identify the one-particle Hilbert space $\mathcal{H}_1$ with

$$\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1}) \oplus \overline{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})} \,, \quad \text{(VII.13)}$$

where the second factor in the direct sum denotes the Hilbert space that is complex-conjugate[5],[6] to $\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$; and,

ii.) Define two bounded linear maps $\kappa_\pm \colon \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1}) \to \mathcal{H}_1$ (see [1]) by

$$\kappa_\pm \colon \alpha \mapsto \Big((\mathbb{1} + \rho_\pm)^{1/2} \alpha, \overline{\rho_\pm^{1/2} \alpha} \Big) \,. \quad \text{(VII.14)}$$

This will allow us to define the quantization maps $\wedge_\pm$. Before we do so, we mention the following property of $\kappa_\pm$.

Proposition VII.4. *Define* $\ell_\pm = \mu_\pm \oplus (-\mu_\mp)$*. The maps*

$$\mathbb{R} \ni s \mapsto \kappa_\pm(\mathrm{e}^{is\mu_\pm} \alpha) = \mathrm{e}^{is\ell_\pm} \kappa_\pm(\alpha) \,,$$

extend analytically to the strip $\{ s + it \in \mathbb{C} \mid 0 < t < \frac{\beta}{2} \}$*. Moreover, they satisfy* one-particle β-KMS conditions*: For* $\alpha, \alpha' \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$ *and* $s \in \mathbb{R}$*, we have*

$$\langle \kappa_\pm(\alpha), \kappa_\pm(\mathrm{e}^{(is-\beta)\ell_\pm} \alpha') \rangle_{\mathcal{H}_1} = \langle \kappa(\mathrm{e}^{is\ell_\pm} \alpha'), \kappa(\alpha) \rangle_{\mathcal{H}_1} \,. \quad \text{(VII.15)}$$

We can now define two one-particle quantization maps $\wedge_\pm \colon \mathcal{K}_{+,0} \mapsto \mathcal{H}_1$ by setting

$$\widehat{f}^{\,\pm}(\vec{x}) = \int_0^{\beta/2} \mathrm{e}^{-t\ell_\pm} \kappa_\pm(f_t(\vec{x})) \, dt \,. \quad \text{(VII.16)}$$

[5]Let $\mathfrak{h}$ be a complex Hilbert space of functions. Then the *conjugate Hilbert space* $\overline{\mathfrak{h}}$ is defined as the Hilbert space $\mathfrak{h}$ with the complex structure $-i$ and the inner product $\langle h_1, h_2 \rangle_{\overline{\mathfrak{h}}} = \langle h_2, h_1 \rangle_{\mathfrak{h}}$. There is a natural $\mathbb{C}$-*linear* map $\mathfrak{h} \mapsto \overline{\mathfrak{h}}$ given by $f \mapsto \overline{f}$, the complex-conjugate of the function f.

[6]In Dirac's notation, if $|f\rangle \in \mathfrak{h}$, then $\langle g| \in \overline{\mathfrak{h}}$. Clearly, $|\lambda f\rangle = \lambda |f\rangle$ and $\langle \mu g| = \overline{\mu} \langle g|$ for $\lambda, \mu \in \mathbb{C}$. Thus, the map $|f\rangle \mapsto \langle \overline{f}|$ is *linear*.

$\mathcal{K}_{+,0} \subset \mathcal{K}_+$ is the dense subset defined as the linear span of $C_0^\infty(S^1_+) \times C_0^\infty(\mathbb{R}^{d-1})$.

Proposition VII.5. *The maps $\wedge_\pm$ agree with the Osterwalder-Schrader quantizations defined by ϑD^c and $D^c\vartheta$. Namely, for $f, g \in \mathcal{K}_{+,0}$,*

$$\langle f, \vartheta D^c g\rangle_{\mathcal{K}} = \left\langle \widehat{f}^{\,+}, \widehat{g}^{\,+}\right\rangle_{\mathcal{H}_1}$$
$$\langle f, D^c\vartheta g\rangle_{\mathcal{K}} = \left\langle \widehat{f}^{\,-}, \widehat{g}^{\,-}\right\rangle_{\mathcal{H}_1}.$$

They extend by continuity to sharp-time test functions $f = \delta_s \otimes \alpha$, with $s \in [0, \frac{\beta}{2}]$ and $\alpha \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{(d-1)})$. Explicitly,

$$\left\langle \widehat{\delta_s \otimes \alpha}^{\,\pm}, \widehat{\delta_{s'} \otimes \alpha'}^{\,\pm}\right\rangle_{\mathcal{H}_1} = \begin{cases} \langle \alpha, (\vartheta D^c)(s,s')\,\alpha'\rangle_{\mathfrak{H}_{-\frac{1}{2}}}\ , & \text{in case } + \ , \\ \langle \alpha, \overline{(\vartheta D^c)(s,s')}\,\alpha'\rangle_{\mathfrak{H}_{-\frac{1}{2}}}\ , & \text{in case } - \ . \end{cases}$$

Proof. We have

$$\left\langle f, \vartheta D^c g\right\rangle_{\mathcal{K}} = \left\langle \int_0^{\beta/2} \mathrm{e}^{-t\boldsymbol{\ell}_+}\kappa_+(f_t)\,dt\ , \int_0^{\beta/2} \mathrm{e}^{-t'\boldsymbol{\ell}_+}\kappa_+(g_{t'})\,dt'\right\rangle_{\mathcal{H}_1}$$
$$\left\langle f, D^c\vartheta g\right\rangle_{\mathcal{K}} = \left\langle \int_0^{\beta/2} \mathrm{e}^{-t\boldsymbol{\ell}_-}\kappa_-(f_t)\,dt\ , \int_0^{\beta/2} \mathrm{e}^{-t'\boldsymbol{\ell}_-}\kappa_-(g_{t'})\,dt'\right\rangle_{\mathcal{H}_1}.$$

This verifies the first two statements. Next, note that $\vartheta K^c = K^c\vartheta$ and $\vartheta L^c = -L^c\vartheta$, so $\vartheta D^{c\,\frac{1}{2}} = (D^{c\,\frac{1}{2}})^*\vartheta$. Hence,

$$\langle f, \vartheta D^c f\rangle_{\mathcal{K}} = \langle f, \vartheta D^{c\,\frac{1}{2}} D^{c\,\frac{1}{2}} f\rangle_{\mathcal{K}} = \langle D^{c\,\frac{1}{2}} f, \vartheta D^{c\,\frac{1}{2}} f\rangle_{\mathcal{K}}\ .$$

As ϑ is unitary, one can use the Schwarz inequality in $\mathcal{K}$, as well as inequality (VII.10) of Proposition VII.2. Moreover, we have seen in (VII.11) that there is a constant $M < \infty$ such that for all $f \in \mathcal{K}_{+,0}$,

$$\|\widehat{f}^{\,\pm}\|_{\mathcal{H}_1} \leqslant M \|f\|_{\mathfrak{H}_{-1}(\boldsymbol{X})}\ . \tag{VII.17}$$

As $\mathcal{K}_{+,0}$ is dense in $\mathfrak{H}_{-1}(\boldsymbol{X})$, (VII.17) ensures that the maps $\wedge_\pm$ extend by continuity to maps from $\mathfrak{H}_{-1}(\boldsymbol{X}_+)$ to $\mathcal{H}_1$. For $s, s' \in S^1_+$ fixed, one can interpret $\vartheta D^c(x,x')$ as defining a transformation on the Sobolev space $\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$, namely,

$$(\vartheta D^c)(s,s') = (\mathbb{1} + \rho_+)\mathrm{e}^{-(s+s')\mu_+} + \overline{\rho_+\mathrm{e}^{(s+s')\mu_+}} \tag{VII.18}$$

for $s, s' \in S^1_+$. Similarly, for $s, s' \in S^1_+$,

$$(D^c\vartheta)(s,s') = (\mathbb{1} + \rho_-)\mathrm{e}^{-(s+s')\mu_-} + \overline{\mathrm{e}^{(s+s')\mu_-}\rho_-}\ . \tag{VII.19}$$

Also, $\overline{\vartheta D^c(s,s')} = D^c\vartheta(s,s')$. Thus, (VII.17) follows from (VII.18) and (VII.19). ∎

We summarize our results in a commutative diagram that relates quantization, compactification, and the Araki-Woods maps $\kappa_\pm$:

$$\begin{array}{ccc}
\mathfrak{H}_{-1}(\mathbb{R}^d_+) & \xrightarrow{\text{compactification}} & \mathfrak{H}_{-1}(S^1_+ \times \mathbb{R}^{d-1}) \\
\big\downarrow \wedge_\pm & & \big\downarrow \wedge_\pm \\
\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1}) & \xrightarrow{\kappa_\pm} & \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1}) \oplus \overline{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})}
\end{array} .$$

Remark VII.6. The maps $\kappa_\pm$ (and thus also the maps $\wedge_\pm$) are linear. We might as well define two anti-linear maps $\kappa'_\pm \colon \mathcal{K}_{+,0} \mapsto \mathcal{H}_1$ by setting

$$\kappa'_\pm(\alpha) = \left(\rho_\pm^{1/2}\alpha, \overline{(\mathbb{1} + \rho_\pm)^{1/2}\alpha} \right) . \tag{VII.20}$$

For $\alpha \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$, the map $\mathbb{R} \ni s \mapsto \kappa'_\pm(\mathrm{e}^{is\mu_\pm}\alpha) = \mathrm{e}^{is\ell_\pm}\kappa'_\pm(\alpha)$ extends analytically to the strip $\{s + it \in \mathbb{C} \mid -\frac{\beta}{2} < t < 0\}$. This leads to two anti-linear quantization maps $\#_\pm$, namely,

$$f^{\#_\pm}(\vec{x}) = \int_0^{-\beta/2} \mathrm{e}^{-t\ell_\pm}\kappa'_\pm(f_t(\vec{x}))\, dt .$$

VII.4. **Time Translation and its Unbounded Quantization.** Let $T(s)$ denote the unitary time translation group on $\mathcal{K} = L_2(\boldsymbol{X})$, by

$$T(s)f_t = f_{t-s} , \quad \text{or} \quad (T(s)f)(t, \vec{x}) = f(t - s, \vec{x}) .$$

The periodicity of time causes a problem for the quantization of $T(s)$. The function $T(s)f_t$, for $0 \leqslant s$, is supported at positive-time (*i.e.*, in the time-interval $[0, \frac{\beta}{2}]$), only if f_t is supported in the time-interval $[0, \frac{\beta}{2} - s]$. The domain of $\widehat{T(s)}^{\pm}$ does not include all of $\widehat{\mathcal{K}}^\pm_+$, and consequently $\widehat{T(s)}^{\pm}$ must be unbounded. Recall $\ell_\pm = \mu_\pm \oplus (-\mu_\mp)$.

Proposition VII.7. *Let $s \in (0, \frac{\beta}{2})$ and let $\mathcal{D}_s$ be the linear span of $\delta_t \otimes \alpha$ for $t \in [0, \frac{\beta}{2} - s]$. The quantizations $\widehat{T(s)}^{\pm}$ of $T(s)$ with domains $\widehat{\mathcal{D}}^\pm_s$ have self-adjoint closures on $\mathcal{H}_1$. Explicitly, these are given by*

$$\widehat{T(s)}^{\pm} = \mathrm{e}^{-s\boldsymbol{\ell}_\pm}. \tag{VII.21}$$

The spectrum of ℓ_+ and ℓ_- is $\mathbb{R} \setminus (-M, M)$, where $M = m\sqrt{1 - \vec{v}^{\,2}}$.

Remark VII.8. Returning from the rescaled time to the proper time amounts to replacing $\ell_\pm$ by $(1 - \vec{v}^{\,2})^{-1/2}\ell_\pm$. The latter has spectrum $\mathbb{R} \setminus (-m, m)$.

Proof. The fact that the quantizations of $\mathcal{D}_s$ are dense in $\mathcal{H}_1$ follows from Proposition VII.9. The matrix elements of $\widehat{T(s)}^{\pm}$ in sharp-time vectors follow from Proposition VII.5. If one restricts α to have its Fourier transform supported on a fixed compact domain, then both $\mu_\pm$, as well as $\widehat{T(s)}^{\pm}$, are bounded operators on such a subspace. Such subspaces of functions are dense in $\mathfrak{H}_{-\frac{1}{2}}$, so both $\widehat{T(s)}^{+}$ and $\widehat{T(s)}^{-}$ are essentially self-adjoint. The spectral properties follow from those of $\mu_\pm$ established in §III. □

Proposition VII.9. *Let $\mathcal{D}_{s_1,s_2}$ denote the linear span of generalized functions of the form $\delta_s \otimes \alpha$, with $\alpha \in \mathfrak{H}_{-\frac{1}{2}}$ and $s = s_1$ or $s = s_2$ for $s_1 \neq s_2$. Then $\widehat{\mathcal{D}}^{\pm}_{s_1,s_2}$ are dense in $\mathcal{H}_1$.*

Proof. We show that the range under quantization of two distinct sharp times s_1, s_2 gives a core of $\mathcal{H}_1$. On the contrary, suppose there exists a unit vector $\chi \in \mathcal{H}_1$ with components χ_1, χ_2, which is orthogonal to all vectors of the form $\widehat{\delta_{s_j} \otimes \alpha}^{+}$ for $j = 1$ and $j = 2$. According to this assumption,

$$\begin{aligned}\left\langle \chi, \widehat{\delta_s \otimes \alpha}^{+} \right\rangle_{\mathcal{H}_1} &= \langle \chi_1, (\mathbb{1} + \rho_+)^{1/2} \mathrm{e}^{-s\mu_+} \alpha \rangle_{\mathfrak{H}_{-\frac{1}{2}}} \\ &\quad + \left\langle \chi_2, (\mathbb{1} + \rho_-)^{1/2} \mathrm{e}^{-(\frac{\beta}{2} - s)\mu_-} \alpha \right\rangle_{\mathfrak{H}_{-\frac{1}{2}}} \\ &= 0\end{aligned}$$

for all $\alpha \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$ and for $s = s_1, s_2$. Taking adjoints in the inner products, as $\mu_\pm^* = \mu_\mp$ and $\rho_\pm^* = \rho_\mp$, we infer

$$(\mathbb{1} + \rho_-)^{1/2} \mathrm{e}^{-s_j \mu_-} \chi_1 = -(\mathbb{1} + \rho_+)^{1/2} \mathrm{e}^{-(\frac{\beta}{2} - s_j)\mu_+} \chi_2 \tag{VII.22}$$

for $j = 1, 2$. There is no loss of generality to assume $s_1 < s_2$, so $\mathrm{e}^{-(s_2 - s_1)\mu_-}$ is bounded. Thus, we arrive at the following system of equations

$$\begin{aligned}(\mathbb{1} + \rho_-)^{1/2} \mathrm{e}^{-s_2 \mu_-} \chi_1 &= -\mathrm{e}^{-(s_2 - s_1)\mu_-} (\mathbb{1} + \rho_+)^{1/2} \mathrm{e}^{-(\frac{\beta}{2} - s_1)\mu_+} \chi_2 \ , \\ (\mathbb{1} + \rho_-)^{1/2} \mathrm{e}^{-s_2 \mu_-} \chi_1 &= -(\mathbb{1} + \rho_+)^{1/2} \mathrm{e}^{-(\frac{\beta}{2} - s_2)\mu_+} \chi_2 \ .\end{aligned}$$

Eliminating χ_1 yields

$$\mathrm{e}^{-(s_2 - s_1)\mu_-} (\mathbb{1} + \rho_+)^{1/2} \mathrm{e}^{-(\frac{\beta}{2} - s_1)\mu_+} \chi_2 = (\mathbb{1} + \rho_+)^{1/2} \mathrm{e}^{-(\frac{\beta}{2} - s_2)\mu_+} \chi_2 \ .$$

Note that μ_+, μ_-, ρ_+ and ρ_- all commute. Thus, multiplying both sides with $(\mathbb{1} + \rho_+)^{-1/2} \mathrm{e}^{(\frac{\beta}{2} - s_2)\mu_+}$, we arrive at

$$\chi_2 = \mathrm{e}^{-(s_2 - s_1)(\mu_+ + \mu_-)} \chi_2 = \mathrm{e}^{-2(s_2 - s_1)\mu} \chi_2 \ , \tag{VII.23}$$

where we use $\mu_+ + \mu_- = 2\mu$. But $0 < s_2 - s_1$ and $0 < m \leqslant \mu$, so $\|\mathrm{e}^{-2(s_2 - s_1)\mu}\| < 1$. Thus, (VII.23) can only hold in case $\chi_2 = 0$, and

(VII.22) implies $\chi_1 = 0$, since $(\mathbb{1} + \rho_-)^{1/2} \mathrm{e}^{-s_j \mu_-}$ is not singular. Hence, $\chi \equiv 0$, which contradicts the assumption $\|\chi\| = 1$. □

VII.5. **The Tomita-Takesaki Operators.** We introduce a time-reflection operator θ that leaves $\boldsymbol{X}_\pm$ invariant by

$$\theta \colon (t, \vec{x}) \mapsto (\tfrac{\beta}{2} - t, \vec{x}) \ .$$

Thus, θ reflects the time about $t = \pm\frac{\beta}{4}$, depending on whether t is positive or negative. It follows that

$$\theta L_2(\boldsymbol{X}_\pm) = L_2(\boldsymbol{X}_\pm) \ .$$

Acting on $L_2(\boldsymbol{X})$, or on $L_2(S^1) \times \mathfrak{H}_{-1/2}(\mathbb{R}^{d-1})$, the operator θ is a self-adjoint, real, symmetric, idempotent, and commutes with ϑ,

$$\theta^* = \theta = \overline{\theta} \ , \quad \theta^2 = \mathbb{1} \ , \quad \vartheta\theta = \theta\vartheta \ .$$

Consequently, θ commutes with $\vartheta D^c(t, t')$. The map $f \mapsto \theta \overline{f}$ maps $f_t \mapsto \overline{f_{\frac{\beta}{2} - t}}$, and thus induces an anti-linear involution whose quantization is the Tomita-Takesaki modular conjugation. In order to verify this claim, we define the relevant modular objects. The spaces

$$\mathcal{L}_\pm = \kappa_\pm \big(\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1}) \big)$$

are real subspaces in $\mathcal{H}_1 = \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1}) \oplus \overline{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})}$. Multiplication by $(i \oplus -i)$ preserves the subspaces $\mathcal{L}_\pm$, but multiplication by $\boldsymbol{i} = (i \oplus i)$ does not. Moreover,

i.) $\mathcal{L}_\pm \cap \boldsymbol{i}\mathcal{L}_\pm = \{0\}$;
ii.) $\mathcal{L}_\pm + \boldsymbol{i}\mathcal{L}_\pm$ is dense in $\mathcal{H}_1$.

It is interesting to note that

$$\boldsymbol{i}\mathcal{L}_\pm = \kappa'_\pm \big(\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1}) \big) \ .$$

Eckmann and Osterwalder [7] have shown that, whenever a real subspace of a Hilbert space satisfies *i.)* and *ii.)*, one can define an anti-linear operator $s_\pm$ by setting

$$\begin{aligned} s_\pm : \mathcal{L}_\pm + \boldsymbol{i}\mathcal{L}_\pm &\to \mathcal{L}_\pm + \boldsymbol{i}\mathcal{L}_\pm \\ k + \boldsymbol{i}k' &\mapsto -k + \boldsymbol{i}k'. \end{aligned}$$

The operator $s_\pm$ are closable. The polar decompositions of their closure

$$\overline{s}_\pm = j \delta_\pm^{1/2} \ , \tag{VII.24}$$

define the modular conjugation j, the modular operators $\delta_\pm^{1/2}$, and the one-particle Liouvillian $L_\pm$. In our case, j maps $(f, \overline{g})$ to $(-g, -\overline{f})$. Thus,

$$j \circ \kappa_\pm = -\kappa'_\pm \ , \qquad \text{and} \quad j\mathcal{L}_\pm = \boldsymbol{i}\mathcal{L}_\pm \ . \tag{VII.25}$$

The modular operator $\delta_{\pm}^{1/2}$ is related to the one-particle Liouvillian,

$$\delta_{\pm}^{1/2} = \mathrm{e}^{-\beta \ell_{\pm}/2} \,, \quad \text{with} \quad \ell_{\pm} = \mu_{\pm} \oplus (-\mu_{\mp}) \,. \tag{VII.26}$$

We summarize our results.

Proposition VII.10. *Let $\mathscr{C}$ denote the anti-linear operator of complex conjugation $\mathscr{C} f = \overline{f}$. Then one has the commutative diagram:*

$$\begin{array}{ccc} \mathfrak{H}_{-1}(\boldsymbol{X}_+) & \xrightarrow{\theta\mathscr{C}} & \mathfrak{H}_{-1}(\boldsymbol{X}_+) \\ \big\downarrow \wedge_{\pm} & \overset{\#_{\pm}}{\searrow} & \big\downarrow \wedge_{\pm} \\ \mathcal{H}_1 & \xrightarrow{\;\;j\;\;} & \mathcal{H}_1 \,. \end{array}$$

VII.6. **The Araki-Woods Fock Space.** The one-particle space $\mathcal{H}_1 = \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1}) \oplus \overline{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})}$ gives rise to a Fock space of the form (III.4). The elements of $(h, h') \in \mathcal{H}_1$ have two components, namely $h \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$ and $h' \in \overline{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})}$. The Fock *annihilation operator*

$$a(h \oplus h') = \Big(\int_{\mathbb{R}^{d-1}} h(\vec{x}) a(\vec{x}) d\vec{x} \Big) \oplus \Big(\int_{\mathbb{R}^{d-1}} h'(\vec{x}')\, a(\vec{x}')\, d\vec{x}' \Big)$$

(which is actually a densely-defined bilinear form on $\mathcal{H} \times \mathcal{H}$) has non-vanishing matrix elements from $\mathcal{H}_n$ to $\mathcal{H}_{n-1}$. In the Fourier representation it acts as

$$\begin{aligned} &(a(\vec{k} \oplus \vec{k}') f)_{n-1}(\vec{k}_1 \oplus \vec{k}_1', \ldots, \vec{k}_{n-1} \oplus \vec{k}_{n-1}') \\ &\qquad = \sqrt{n}\, f_n(\vec{k} \oplus \vec{k}', \vec{k}_1 \oplus \vec{k}_1', \ldots, \vec{k}_{n-1} \oplus \vec{k}_{n-1}') \,. \end{aligned}$$

and satisfies $[a(\vec{k}_1 \oplus \vec{k}_1'), a(\vec{k}_2 \oplus \vec{k}_2')] = 0$. The adjoint creation form $a(\vec{k} \oplus \vec{k}')^*$ satisfies the usual canonical relations, namely,

$$[a(\vec{k}_1 \oplus \vec{k}_1'), a^*(\vec{k}_2 \oplus \vec{k}_2')] = \delta(\vec{k}_1 - \vec{k}_2) \oplus \delta(\vec{k}_1' - \vec{k}_2') \,.$$

The hermitian time-zero field $\varphi(\vec{x} \oplus \vec{x}')$ on $\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$ is defined as

$$\begin{aligned} \varphi(\vec{x} \oplus \vec{x}') = (2\pi)^{-\frac{d-1}{2}} &\int \frac{d\vec{k}}{\sqrt{2\mu(\vec{k})}} \Big(a^*(\vec{k}) + a(-\vec{k}) \Big) \mathrm{e}^{-i\vec{k}\cdot\vec{x}} \\ &\oplus (2\pi)^{-\frac{d-1}{2}} \int \frac{d\vec{k}'}{\sqrt{2\mu(\vec{k}')}} \Big(a^*(\vec{k}') + a(-\vec{k}') \Big) \mathrm{e}^{-i\vec{k}'\cdot\vec{x}'} \,. \end{aligned}$$

The Liouvillean $L_{\pm}$, the momentum operator $\vec{P}$, the modular conjugation J and the Tomita operator S act on $\mathcal{H}_1$ as $\ell_{\pm}$, $\vec{p}$, j and s, respectively. Note that

$$\Omega_0 \in \mathcal{D}(L_\pm) \quad \text{and} \quad L_\pm \Omega_0 = 0 \; .$$

The spectrum of $L_\pm$ is the real line $\mathbb{R}$, and its zero eigenvalue is simple. The latter follows from the gap in the spectrum of $\ell_\pm$ [23, Theorems 1a & 1b].

VII.7. **Quantization of Field Operators.** The quantization map (VII.16) for vectors fixes the quantization map for the field operators, as we require that

$$\widehat{\Phi(f)}^{\pm} \widehat{\Omega}^{\pm} = \widehat{\Phi(f)\Omega}^{\pm} \quad \text{for} \quad \Omega \in \mathcal{D}(\Phi(f)) \cap \mathcal{E}_\pm \; . \tag{VII.27}$$

This implies that

$$\begin{aligned} \langle (\widehat{\Phi(f)}^{\pm})^n \, \Omega_0, (\widehat{\Phi(f)}^{\pm})^n \, \Omega_0 \rangle_{\mathcal{H}} &= \langle \Phi(f)^n \Omega_0^E, \Theta_\pm \Phi(f)^n \Omega_0^E \rangle_{\mathcal{E}} \\ &= (2n-1)!! \langle \widehat{f}^{\pm}, \widehat{f}^{\pm} \rangle_{\mathcal{H}_1}^n \; , \end{aligned} \tag{VII.28}$$

where $\Theta_\pm$ are the reflections associated to ϑD^c and $D^c \vartheta$, respectively. As before, $\mathcal{H}_1 = \mathfrak{H}_{-\frac{1}{2}} \oplus \overline{\mathfrak{H}_{-\frac{1}{2}}}$ and

$$\langle \widehat{f}^{\pm}, \widehat{f}^{\pm} \rangle_{\mathcal{H}_1} = \Big\| \, | \int_0^{\beta/2} \mathrm{e}^{-t\mu_\pm} (\mathbb{1} + \rho_\pm)^{1/2} f_t \, dt \; , \int_0^{\beta/2} \mathrm{e}^{t\mu_\pm} \rho_\pm^{\frac{1}{2}} \, \overline{f_t} \, dt \rangle \, \Big\|_{\mathcal{H}_1}^2 \; .$$

The Gaussian nature of the Fock space $\mathcal{E}$ together with Proposition VII.5 imply that (VII.28) is satisfied if we set

$$\widehat{\Phi(f)}^{\pm} = \varphi(\widehat{f}^{\pm}) \; , \qquad f \in \mathcal{E}_1 \cap \mathcal{E}_\pm \; .$$

Note that $\widehat{f}^{\pm}$ has two components, and therefore $\widehat{f}^{\pm}$ can be viewed as a function on $\mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$. For certain sharp-time functions, f and g, the quantized field operators take a special form.

Proposition VII.11. *For $f = \delta \otimes \alpha$ with $\alpha \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$, we have*

$$\widehat{\Phi(f)}^{\pm} = \widehat{\Phi(0,\alpha)}^{\pm} = \varphi\big(\kappa_\pm(\alpha)\big) \; . \tag{VII.29}$$

Moreover, for $g = \delta_{\frac{\beta}{2}} \otimes \alpha$, where $\delta_{\frac{\beta}{2}}(t) = \delta(t - \frac{\beta}{2})$ denotes the shifted Dirac delta function, we have

$$\widehat{\Phi(g)}^{\pm} = \widehat{\Phi(\tfrac{\beta}{2},\alpha)}^{\pm} = \varphi\big(\kappa'_\pm(\alpha)\big) \; . \tag{VII.30}$$

Proof. The identity (VII.29) follows from (VII.27) and (VII.16). The second identity, (VII.30), follows from

$$\begin{aligned}\widehat{g}^{\pm} &= \mathrm{e}^{-\beta\ell_\pm/2}\kappa_\pm(\alpha)\\ &= \Big(\mathrm{e}^{-\beta\mu_\pm/2}(\mathbb{1}+\rho_\pm)^{1/2}\alpha, \overline{\mathrm{e}^{\beta\mu_\pm/2}\rho_\pm^{1/2}\alpha}\Big)\\ &= \Big(\rho_\pm^{1/2}\alpha, \overline{(\mathbb{1}+\rho_\pm)^{1/2}\alpha}\Big) = \kappa'_\pm(\alpha)\,.\end{aligned}$$

□

The bounded functions of the time-zero fields generate abelian von Neumann algebras

$$\mathcal{U}_0^{\pm} = \{\widehat{\Phi(0,\alpha)}^{\pm} \mid \alpha \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})\}''\,.$$

Similarly, the bounded functions of the time-$\frac{\beta}{2}$ fields generate another two abelian von Neumann algebras,

$$\mathcal{U}_{\frac{\beta}{2}}^{\pm} = \{\widehat{\Phi(\frac{\beta}{2},\alpha)}^{\pm} \mid \alpha \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})\}''\,.$$

Not only do they commute with each other, but also their time translates commute with each other. For $\alpha \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$ and $s \in \mathbb{R}$,

$$\begin{aligned}\varphi_\pm(s,\alpha) &= \mathrm{e}^{isL_\pm}\varphi(\kappa_\pm(\alpha))\mathrm{e}^{-isL_\pm}\,,\\ \varphi'_\pm(s,\alpha) &= \mathrm{e}^{isL_\pm}\varphi(\kappa'_\pm(\alpha))\mathrm{e}^{-isL_\pm}\,.\end{aligned}$$

We will also use $\varphi_\pm(\alpha) = \varphi_\pm(0,\alpha)$ and $\varphi'_\pm(\alpha) = \varphi'_\pm(0,\alpha)$.

Proposition VII.12. *For $\alpha, \alpha' \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$ and $s, s' \in \mathbb{R}$,*

$$\big[\,\varphi_\pm\big(-s,\overline{\alpha}\big), \varphi'_\pm\big(s',\alpha'\big)\big] = 0\,.$$

In particular, for $\alpha, \alpha' \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$,

$$\big[\,\widehat{\Phi(0,\alpha)}^{\pm}, \widehat{\Phi(\tfrac{\beta}{2},\alpha')}^{\pm}\,\big] = 0.$$

Proof. We compute

$$\begin{aligned}&\langle \mathrm{e}^{isL_\pm}\varphi\big(\kappa_\pm(\alpha)\big)\,\Omega_0, \mathrm{e}^{itL_\pm}\varphi\big(\kappa'_\pm(\alpha')\big)\,\Omega_0\rangle_{\mathcal{H}}\\ &\quad= \langle \mathrm{e}^{is\ell_\pm}\kappa_\pm(\alpha), \mathrm{e}^{it\ell_\pm}\kappa'_\pm(\alpha')\rangle_{\mathcal{H}_1}\\ &\quad= \langle \mathrm{e}^{is\mu_\pm}\alpha, \rho_\pm^{1/2}(\mathbb{1}+\rho_\pm)^{1/2}, \mathrm{e}^{it\mu_\pm}\alpha'\rangle_{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})}\\ &\qquad+ \langle \mathrm{e}^{-it\mu_\pm}\overline{\alpha}', \rho_\pm^{1/2}(\mathbb{1}+\rho_\pm)^{1/2}\mathrm{e}^{-is\mu_\pm}\overline{\alpha}\rangle_{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})}\,.\end{aligned}$$

Similarly,

$$\begin{aligned}
&\langle \mathrm{e}^{-itL_\pm}\varphi\big(\kappa'_\pm(\overline{\alpha}')\big)\,\Omega_0, \mathrm{e}^{-isL_\pm}\varphi\big(\kappa_\pm(\overline{\alpha})\big)\,\Omega_0\rangle_{\mathcal{H}} \\
&\quad = \langle \mathrm{e}^{-it\ell_\pm}\kappa'_\pm(\overline{\alpha}'), \mathrm{e}^{-is\ell_\pm}\kappa_\pm(\overline{\alpha})\rangle_{\mathcal{H}_1} \\
&\quad = \langle \mathrm{e}^{-it\mu_\pm}\overline{\alpha}', \rho_\pm^{1/2}(\mathbb{1}+\rho_\pm)^{1/2}, \mathrm{e}^{-is\mu_\pm}\overline{\alpha}\rangle_{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})} \\
&\qquad + \langle \mathrm{e}^{is\mu_\pm}\overline{\alpha}, \rho_\pm^{1/2}(\mathbb{1}+\rho_\pm)^{1/2}\mathrm{e}^{it\mu_\pm}\alpha'\rangle_{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})}.
\end{aligned}$$

Therefore, the expectation of the commutator vanishes. As the commutator is a scalar, it must equal zero. The second commutator claimed to vanish is a special case, as the fields on the boundary can obtained from (VII.30). □

That is, the adjoint action of the unitary group $\mathrm{e}^{itL_\pm}$, $t \in \mathbb{R}$, on $\mathcal{U}_0^\pm$ and $\mathcal{U}_{\frac{\beta}{2}}^\pm$ gives rise to two commuting non-abelian algebras,

$$\mathcal{R}_0^\pm = \{\varphi_\pm(s,\alpha) \mid s \in \mathbb{R}, \alpha \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})\}'' \tag{VII.31}$$

and

$$\mathcal{R}_{\frac{\beta}{2}}^\pm = \{\varphi'_\pm(s,\alpha) \mid s \in \mathbb{R}, \alpha \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})\}''\,, \tag{VII.32}$$

respectively.

Proposition VII.13. *Let* $\alpha_i \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$ *and* $0 \leqslant s_i$, $1 \leqslant i \leqslant n$. *Moreover, assume that* $\sum_{j=1}^n s_j \leqslant \frac{\beta}{2}$. *Then*

$$\mathrm{e}^{-s_{n-1}L_\pm}\varphi_\pm(\alpha_{n-1})\cdots \mathrm{e}^{-s_1L_\pm}\varphi_\pm(\alpha_1)\Omega_0 \in \mathcal{D}\big(\varphi_\pm(\alpha_n)\big)$$

and

$$\varphi_\pm(\alpha_n)\mathrm{e}^{-s_{n-1}L_\pm}\varphi_\pm(\alpha_{n-1})\cdots \mathrm{e}^{-s_1L_\pm}\varphi_\pm(\alpha_1)\Omega_0 \in \mathcal{D}\big(\mathrm{e}^{-s_nL_\pm}\big)\,.$$

Furthermore, the linear span of such vectors is dense in $\mathcal{H}$ *and*

$$\begin{aligned}
&\mathrm{e}^{-s_nL_\pm}\varphi_\pm(\alpha_n)\mathrm{e}^{-(s_{n-1}+s_n)L_\pm}\varphi_\pm(\alpha_{n-1})\ldots \mathrm{e}^{-(s_1-s_2)L_\pm}\varphi_\pm(\alpha_1)\Omega_0 \\
&\quad = \Big(\Phi(s_n,\alpha_n)\Phi(s_{n-1},\alpha_{n-1})\cdots\Phi(s_1,\alpha_1)\Omega_0^E\Big)^{\wedge_\pm}\,.
\end{aligned} \tag{VII.33}$$

Proof. Let $\boldsymbol{T}(s)$ be the second quantization of the unitary time translation $T(s)$ introduced in Section VII.4. Its quantization is the second quantization of $\mathrm{e}^{-s\ell_\pm}$, *i.e.*,

$$\widehat{\boldsymbol{T}(s)}^{\,\pm} = \mathrm{e}^{-sL_\pm}\,.$$

It follows that for $0 \leqslant s_i$, $1 \leqslant i \leqslant n$, and $\sum_{j=1}^n s_j \leqslant \frac{\beta}{2}$,

$$\boldsymbol{T}(s_{n-1})\Phi(0,\alpha_{n-1})\cdots\boldsymbol{T}(s_1)\Phi(0,\alpha_1)\Omega_0^E \in \mathcal{D}\left(\Phi(0,\alpha_{\alpha_n})\right)$$

and

$$\Phi(0,\alpha_n)\boldsymbol{T}(s_{n-1})\Phi(0,\alpha_{n-1})\cdots\boldsymbol{T}(s_1)\Phi(0,\alpha_1)\Omega_0^E \in \mathcal{D}\left(\boldsymbol{T}(s_n)\right) .$$

The results now follow from (VII.33). The fact that the linear span of such vectors is dense in $\mathcal{H}$ is a consequence of Proposition VII.9. □

Proposition VII.14. *Let $\alpha_i \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$, for $1 \leqslant i \leqslant n$. If $0 \leqslant s_1 \leqslant \cdots \leqslant s_k \leqslant \frac{\beta}{2} \leqslant s_{k+1} \leqslant \ldots \leqslant s_n \leqslant \beta$, then*

$$\begin{aligned}&\Big\langle \Omega_0^E, \prod_{j=1}^n \Phi(s_j,\alpha_j)\Omega_0^E \Big\rangle \qquad \text{(VII.34)}\\ &= \Big\langle \mathrm{e}^{(s_n-\beta)L_\pm}\varphi_\pm(\overline{\alpha}_n)\mathrm{e}^{(s_{n-1}-s_n)L_\pm}\varphi_\pm(\overline{\alpha}_{n-1})\cdots\mathrm{e}^{(s_{k+1}-s_{k+2})L_\pm}\varphi_\pm(\overline{\alpha}_{k+1})\Omega_0 ,\\ &\qquad \mathrm{e}^{-s_1L_\pm}\varphi_\pm(\alpha_1)\mathrm{e}^{(s_1-s_2)L_\pm}\varphi_\pm(\alpha_2)\cdots\mathrm{e}^{(s_{k-1}-s_k)L_\pm}\varphi_\pm(\alpha_k)\Omega_0 \Big\rangle .\end{aligned}$$

Moreover,

$$\left\|\mathrm{e}^{-(\beta/2)L_\pm}\varphi_\pm(\alpha_n)\cdots\varphi_\pm(\alpha_1)\Omega_0\right\|_{\mathcal{H}} = \left\|\varphi_\pm(\alpha_n)\cdots\varphi_\pm(\alpha_1)\Omega_0\right\|_{\mathcal{H}} .$$

Proof. If $0 \leqslant s_1 \leqslant \ldots \leqslant s_k \leqslant \frac{\beta}{2}$ and $\frac{\beta}{2} \leqslant s_{k+1} \leqslant \ldots \leqslant s_n \leqslant \beta$, then according to Proposition VII.13 the right hand side in (VII.34) is well-defined and equals

$$\begin{aligned}&\big\langle (\Phi(\beta-s_n,\overline{\alpha}_n)\cdots\Phi(\beta-s_{k+1},\overline{\alpha}_{k+1})\Omega_0^E\big)^{\wedge\pm},\\ &\qquad\qquad \big(\Phi(s_k,\alpha_k)\cdots\Phi(s_1,\alpha_1)\Omega_0^E\big)^{\wedge\pm}\big\rangle_{\mathcal{H}}\\ &= \Big\langle \Big(\prod_{j=k+1}^n \Phi(\beta-s_j,\overline{\alpha}_j)\Big)\Omega_0^E, \Theta_\pm\Big(\prod_{j=1}^k \Phi(s_j,\alpha_j)\Big)\Omega_0^E\Big\rangle_{\mathcal{E}}\\ &= \Big\langle \Big(\boldsymbol{T}(\beta)\prod_{j=k+1}^n \Phi(-s_j,\overline{\alpha}_j)\Big)\Omega_0^E, \Theta_\pm\Big(\prod_{j=1}^k \Phi(s_j,\alpha_j)\Big)\Big\rangle_{\mathcal{E}}\\ &= \Big\langle \Big(\prod_{j=k+1}^n \Phi(-s_j,\overline{\alpha}_j)\Big)\Omega_0^E, \Theta_\pm\Big(\prod_{j=1}^k \Phi(s_j,\alpha_j)\Big)\Big\rangle_{\mathcal{E}}\\ &= \big\langle \Omega_0^E, \prod_{j=1}^n \Phi(s_j,\alpha_j)\Omega_0^E\big\rangle_{\mathcal{E}} .\end{aligned}$$

We made use of $\boldsymbol{T}(\beta)=1$, which holds by periodicity. By Proposition VII.13 we have

$$\varphi_\pm(\alpha_n)\varphi_\pm(\alpha_{n-1})\cdots\varphi_\pm(\alpha_1)\Omega_0 \in \mathcal{D}\big(\mathrm{e}^{-\beta L_\pm/2}\big) .$$

Now

$$\begin{aligned}
&\left\|e^{-\beta L_\pm/2}\varphi_\pm(\alpha_n)\varphi_\pm(\alpha_{n-1})\cdots\varphi_\pm(\alpha_1)\Omega_0\right\|_{\mathcal{H}}^2 \\
&\quad = \left\|\Big(\boldsymbol{T}(\beta/2)\Phi(0,\alpha_n)\cdots\Phi(0,\alpha_1)\Omega_0^E\Big)^{\wedge_\pm}\right\|_{\mathcal{H}}^2 \\
&\quad = \big\langle \boldsymbol{T}(\beta/2)\Phi(0,\alpha_n)\cdots\Phi(0,\alpha_1)\Omega_0^E\,,\,\Theta_\pm\boldsymbol{T}(\beta/2)\,\Phi(0,\alpha_n)\cdots\Phi(0,\alpha_1)\Omega_0^E\big\rangle_{\mathcal{E}} \\
&\quad = \big\langle \Phi(0,\alpha_n)\cdots\Phi(0,\alpha_1)\Omega_0^E\,,\,\boldsymbol{T}(-\beta/2)\,\Theta_\pm\boldsymbol{T}(\beta/2)\,\Phi(0,\alpha_n)\cdots\Phi(0,\alpha_1)\big\rangle_{\mathcal{E}} \\
&\quad = \big\langle \Phi(0,\alpha_n)\cdots\Phi(0,h_1)\Omega_0^E\,,\,\Theta_\pm\boldsymbol{T}(\beta)\,\Phi(0,\alpha_n)\cdots\Phi(0,h_1)\Omega_0^E\big\rangle \\
&\quad = \left\|\big(\Phi(0,\alpha_n)\cdots\Phi(0,\alpha_1)\Omega_0^E\big)^{\wedge_\pm}\right\|_{\mathcal{H}}^2 \\
&\quad = \|\varphi_\pm(\alpha_n)\varphi_\pm(\alpha_{n-1})\cdots\varphi_\pm(\alpha_1)\Omega_0\|_{\mathcal{H}}^2\,,
\end{aligned}$$

again using $\boldsymbol{T}(\beta) = 1$. □

Proposition VII.15. (Special case of Theorem 2.5.14. in [4])

i.) *The adjoint action of the unitary group* $\{\exp(itL_\pm) \mid t \in \mathbb{R}\}$ *leaves the algebras* $\mathcal{R}_0^\pm$ *and* $\mathcal{R}_{\beta/2}^\pm$ *invariant;*

ii.) *The identity holds,* $J\mathcal{R}_0^\pm J = (\mathcal{R}_0^\pm)' = \mathcal{R}_{\beta/2}^\pm$*; and,*

iii.) *The operator* $S_\pm$ *is closed, its polar decomposition is*

$$S_\pm = J e^{-\beta L_\pm/2}$$

and $S_\pm A\Omega_0 = A^*\Omega_0$ *for all* $A \in \mathcal{R}_0^\pm$*. For* $B \in \mathcal{R}_{\beta/2}^\pm$*, one has* $S_\pm^* B\Omega = B^*\Omega_0$*.*

Proof. Property *i.)* follows from Proposition VII.12. Property *iii.)* follows from the fact that $S_\pm$, J, and $L_\pm$ are the second quantizations of the one particle operators $s_\pm$, j, and $\ell_\pm$ which satisfy (VII.24). Finally, Property *ii.)* follows from (VII.25). □

Corollary VII.16. *The vector* Ω_0 *induces a unique KMS state for the quantum dynamical systems* $(\mathcal{R}_0^\pm, e^{itL_\pm})$ *associated to the one-particle Hamiltonians* $\mu_\pm$ *acting on the Sobolev space* $\mathfrak{H}_{-\frac{1}{2}}(\mathbb{R}^{d-1})$, i.e., *for bounded operators* $A, B \in \mathcal{R}_0$ *the functions*

$$\mathbb{R} \ni t \mapsto F_{A,B}^\pm(t) = \langle \Omega_0, A e^{itL_\pm} B\Omega_0\rangle$$

extend to an analytic functions in the strip $z = t + is$ *with* $0 < s < \beta$*, with continuous boundary values given by*

$$F_{A,B}^\pm(t + i\beta) = \langle \Omega_0, B e^{-itL_\pm} A\Omega_0\rangle\,. \tag{VII.35}$$

This KMS state is Gaussian and its two-point function is

$$\langle \varphi_\pm(0,\alpha)\Omega_0, \varphi_\pm(t',\alpha')\Omega_0\rangle_{\mathcal{H}} = \big\langle \alpha\,,\,\coth(\beta\mu_\pm)\,e^{it'\mu_\pm}\alpha'\big\rangle_{\mathfrak{H}_{-\frac{1}{2}}}\,.$$

The GNS representation associated to the pairs $(\mathcal{R}_0^{\pm}, \langle\Omega_0, \cdot\,\Omega_0\rangle)$ *are the Araki-Woods representations (see Section VII.6).*

Proof. The KMS condition (VII.35) follows directly from Theorem VII.15 (iii). Uniqueness of the KMS state follows from the commutation relations, the KMS condition and $m > 0$. The fact that the GNS representation associated to the pair $(\mathcal{R}_0^{\pm}, \langle\Omega_0, \cdot\,\Omega_0\rangle)$ is the Araki-Woods representation follows from the fact that Ω_0 is cyclic for $\mathcal{R}_0^{\pm}$. □

VIII. Classical Fields on the d-Torus $\boldsymbol{X} = \mathbb{T}^d$

In this section we study periodization of both time and spatial directions. Thus, we are interested in the spacetime for the classical field given by $\boldsymbol{X} = S^1 \times \mathbb{T}^{d-1} = \mathbb{T}^d$ with S^1 a circle of circumference β, and $\mathbb{T}^d$ is the d-dimensional torus. Let, as before, $\Lambda = \prod_{j=1}^{d-1} \ell_j$ denote the spatial volume of the torus $\mathbb{T}^{d-1}$.

VIII.1. **The two-point function.** The fully compactified covariances $D^c_{\pm,\beta,\Lambda} = D^{c*}_{\mp,\beta,\Lambda}$ arise form the covariance $D_{\vec{v}}$ by compactifying both the time and the spatial coordinates. For the kernels this yields

$$\begin{aligned} &D^c_{\pm,\beta,\Lambda}(x-x') \qquad\qquad \text{(VIII.1)}\\ &= \frac{\theta(t'-t)}{\Lambda} \sum_{\vec{k}\in\mathcal{K}_\Lambda} \frac{1}{2\mu(\vec{k})} \left(\frac{\mathrm{e}^{-(t'-t)\mu_\pm(\vec{k})}}{1-\mathrm{e}^{-\beta\mu_\pm(\vec{k})}} + \frac{\mathrm{e}^{-(\beta-(t'-t))\mu_\mp(\vec{k})}}{1-\mathrm{e}^{-\beta\mu_\mp(\vec{k})}} \right) \mathrm{e}^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\\ &+ \frac{\theta(t-t')}{\Lambda} \sum_{\vec{k}\in\mathcal{K}_\Lambda} \frac{1}{2\mu(\vec{k})} \left(\frac{\mathrm{e}^{-(t-t')\mu_\mp(\vec{k})}}{1-\mathrm{e}^{-\beta\mu_\mp(\vec{k})}} + \frac{\mathrm{e}^{-(\beta-(t-t'))\mu_\pm(\vec{k})}}{1-\mathrm{e}^{-\beta\mu_\pm(\vec{k})}} \right) \mathrm{e}^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\ , \end{aligned}$$

where $\theta(t)$ denotes the characteristic function for the half-line $t \geqslant 0$. The kernels

$$D^c_{+,\beta,\Lambda}(x-x') = \overline{D^c_{-,\beta,\Lambda}(x-x')}$$

have smooth limits as each $\ell_j \to \infty$ converging in the limit of infinite volume to the kernels

$$D^c(x-x') = \langle \mathbb{A}\varphi_I^+(x)\varphi_I^+(x')\rangle_{\beta,\pm}$$

introduced to study the time compactification in (VII.4), in the same sense that a Fourier series approximates a Fourier transform. The sum over each coordinate of $\vec{k}$ converges to a Riemann integral, and the limiting kernels coincide with the operators D^c introduced in (VII.4). The corresponding anti-time-ordered two-point functions also converge.

The operators D^c act on $L_2(S^1 \times \mathbb{R}^{d-1})$. Likewise, $D^c_{+,\beta,\Lambda}$ acts on $L_2(S^1 \times \mathbb{T}^{d-1})$. In all cases, these operators equal the inverse of the corresponding differential operators

$$D_{\vec{v}}^{-1} = -\Delta + m^2 + (\nabla_{\vec{x}} \cdot \vec{v})^2 - 2i\tfrac{\partial}{\partial t}(\nabla_{\vec{x}} \cdot \vec{v}) \ ,$$

originally introduced in (III.20) on $\mathbb{R}^d$, but here acting on these (partially or fully) compactified spacetimes.

As such D^c and $D^c_{+,\beta,\Lambda}$ are *doubly temporally reflection-positive.* The operator D^c is *doubly spatially reflection-positive* in the direction $\vec{n}$. The same is true for $D^c_{+,\beta,\Lambda}$ in case $\vec{n}$ lies along a lattice coordinate direction.

VIII.2. **Quantization.** The Osterwalder-Schrader quantization from the d-torus results in only minor changes[7] to the results presented in §VII. The Araki-Woods one-particle Hilbert space $\mathcal{H}_1(\Lambda)$, which arises from Osterwalder-Schrader quantization, is now

$$\mathcal{H}_1(\beta,\Lambda) = \mathfrak{H}_{-\frac{1}{2}}(\mathbb{T}^{d-1}) \oplus \overline{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{T}^{d-1})} \ .$$

The associated Fock space is again of the form (III.4). The elements of $(h,h') \in \mathcal{H}_1$ have two components, namely $h \in \mathfrak{H}_{-\frac{1}{2}}(\mathbb{T}^{d-1})$ and $h' \in \overline{\mathfrak{H}_{-\frac{1}{2}}(\mathbb{T}^{d-1})}$.

The Liouvillean $L_\pm$, the momentum operator $\vec{P}$, the modular conjugation J and the Tomita operator S act on $\mathcal{H}_1(\beta,\Lambda)$ as $\ell_\pm$, $\vec{p}$, j and s. Note that

$$\Omega_0 \in \mathcal{D}(L_\pm) \quad \text{and} \quad L_\pm\Omega_0 = 0$$

still holds. However, the spectrum of $L_\pm$ of is discrete (and symmetric), and the discrete eigenvalue zero is infinitely degenerated.

Remark VIII.1. The spectral properties can be understood by considering Gibbs states on $\mathbb{R} \times \mathbb{T}^{d-1}$; see Remark IV.1 of §IV. Using energy eigenvectors $\Psi_i^\pm$,

$$H_\pm \Psi_i^\pm = E_i^\pm \Psi_i^\pm \qquad \text{with} \qquad E_i \in \mathbb{R}^+ \cup \{0\} \ .$$

The Gibbs density matrix[8] takes the form

$$\frac{\mathrm{e}^{-\beta H_\pm(\Lambda)}}{\operatorname{Tr} \mathrm{e}^{-\beta H_\pm(\Lambda)}} = \frac{\sum_i \mathrm{e}^{-\beta E_i^\pm} \ |\Psi_i^\pm\rangle\langle\Psi_i^\pm|}{\sum_k \mathrm{e}^{-\beta E_k^\pm}}, \qquad \beta > 0 \ . \qquad \text{(VIII.2)}$$

The GNS representation for the state given by the Gibbs density matrix is of the form

$$\mathcal{B}(\mathcal{H}_\Lambda) \ni A \mapsto A \otimes \mathbb{1} \ , \qquad \text{(VIII.3)}$$

[7]The one-particle space $\mathfrak{H}_{-\frac{1}{2}}$ and the Laplace operator have to be adapted to periodic boundary conditions.

[8]The following arguments hold true in the *interacting* case too.

acting on the tensor product $\mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda$ of the Hilbert space $\mathcal{H}_\Lambda$ introduced in (IV.1) with itself. The cyclic vector, *i.e.*, the GNS vector,

$$|\sqrt{\rho_\pm}\rangle = \frac{\sum_i \mathrm{e}^{-\beta E_i^\pm/2}}{\sqrt{\sum_k \mathrm{e}^{-\beta E_k^\pm}}}\ \Psi_i^\pm \otimes \Psi_i^\pm$$

induces the Gibbs state, *i.e.*,

$$\frac{\operatorname{Tr} \mathrm{e}^{-\beta H_\pm(\Lambda)} A}{\operatorname{Tr} \mathrm{e}^{-\beta H_\pm(\Lambda)}} = \langle \sqrt{\rho_\pm}\, , \, (A \otimes \mathbb{1})\sqrt{\rho_\pm}\rangle \ , \qquad A \in \mathcal{B}(\mathcal{H}_\Lambda) \ .$$

The generator of the time evolution in the GNS representation is

$$L_\pm = H_\pm(\Lambda) \otimes \mathbb{1} - \mathbb{1} \otimes H_\pm(\Lambda) \ .$$

In finite volume Λ, the Araki-Woods representation on the Fock space over the one-particle space $\mathcal{H}_1(\beta, \Lambda)$ resulting form the Osterwalder-Schrader quantization is unitarily equivalent to the GNS representation (VIII.3). However, in the infinite volume case discussed in §VII, the von Neumann algebras $\mathcal{R}_0$ and $\mathcal{R}_0' = \mathcal{R}_{\beta/2}$ introduced in (VII.31) and (VII.32), respectively, are both factors of type III and, consequently, their von Neumann tensor product is type III as well, in contrast to $\mathcal{R}_0 \vee \mathcal{R}_0' = \mathcal{B}(\mathcal{H}_\beta)$.

The fact that the spectrum of $L_\pm$ is discrete and symmetric, and the infinite degeneracy of the eigenvalue zero cause a number of problems. However, it is instructive to consider the map $N_\Lambda^\pm \colon \mathcal{B}(\mathcal{H}_\Lambda) \to \mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda$,

$$A \mapsto \mathrm{e}^{-\lambda |L_\pm|}(A \otimes \mathbb{1})\ |\sqrt{\rho_\pm}\rangle \ .$$

A straight forward computation yields

$$\begin{aligned} N_\Lambda^\pm(A) &= \sum_{i,j} \mathrm{e}^{-\lambda |E_i^\pm - E_j^\pm|}(A_{i,j} \otimes \mathbb{1}) \frac{\mathrm{e}^{-\beta E_j^\pm/2}}{\sqrt{\sum_k \mathrm{e}^{-\beta E_k^\pm}}}\ \Psi_j^\pm \otimes \Psi_j^\pm \\ &= \frac{1}{\sqrt{\sum_k \mathrm{e}^{-\beta E_k^\pm}}} \sum_{i,j} \mathrm{e}^{-\lambda |E_i^\pm - E_j^\pm| - \beta E_j^\pm/2} (A_{i,j}^\pm \Psi_j^\pm \otimes \Psi_j^\pm) \ , \end{aligned}$$

where $A_{i,j}^\pm := |\Psi_i^\pm\rangle\langle\Psi_i^\pm|A|\Psi_j^\pm\rangle\langle\Psi_j^\pm|$ is a rank 1 operator. The sum $\sum_{i,j}$ is convergent for $\lambda > 0$; thus, $N_\Lambda^\pm$ is a nuclear map. In fact, $N_\Lambda^\pm$ is nuclear for all $\lambda > 0$ and, consequently, it is an element of all Schatten–von Neumann classes.

Remark VIII.2. The imaginary anti-time-ordered two point function

$$D^c_{\pm,\beta,\Lambda}(x - x') = \langle \mathbb{A} \varphi_I^\pm(x) \varphi_I^\pm(x') \rangle_{\pm,\beta,\Lambda} \qquad \text{(VIII.4)}$$

in the Gibbs state $\langle \cdot \rangle_{\pm,\beta,\Lambda}$ defined by density matrix (VIII.2) agrees with the covariances $D^{c}_{\pm,\beta,\Lambda}$ given in (VIII.1). The proof of this statement relies on two basic facts. Firstly, the *pull-through* identity holds:

$$a(f)\mathrm{e}^{-tH_\pm} = \mathrm{e}^{-tH_\pm} a(\mathrm{e}^{-t\mu_\pm} f) \ .$$

Secondly, cyclicity of the trace, translation invariance of $H_\pm$, and the fact that $H_\pm$ commutes with the number operator N, ensures a symmetry of the expectation of creation and annihilation operators. Let $a(k)^\#$ denote either $a(k)$ or $a(k)^*$. Then

$$\langle a^\#_\pm(k,t) a^{\#'}_\pm(k',t') \rangle_{\beta,\pm,\Lambda} = 0 \ ,$$

unless $k = k'$, as well as one $a(k)^\#$ being a creation operator, while the other $a(k)^{\#'}$ is an annihilation operator. Using these two facts, and the expansion for the time-zero field, we can evaluate the two-point function in closed form. We omit further details.

IX. Some Comments on Partial Wick Rotation

In this paper we use complex classical fields to construct (interacting) quantum fields in finite and infinite volumes both at zero and positive temperatures. These fields describe neutral particles in both vacuum and thermal equilibrium states. We now indicate how the quantum fields we have constructed in this work are related to the quantum fields considered in more traditional approaches.

IX.1. **Flat Space and Spatially Compactified Space.** Let us consider scalar Wightman quantum fields $\varphi(t,\vec{x})$ defined on d-dimensional, Minkowski spacetime, acting on the Hilbert space $\mathcal{H}$. Let $\Omega_0 \in \mathcal{H}$ denote the vacuum vector, and let H and $\vec{P}$ denote the Hamiltonian and momentum operator, respectively. The fields are Poincaré covariant and Ω_0 is Poincaré invariant. Hence, the *Wightman functions* are also Poincaré-invariant functions on spacetime. With (Λ, a) in the Poincaré group, the Wightman function satisfies

$$\begin{aligned}\mathcal{W}^{(n)}(x_1,\ldots,x_n) &= \langle \Omega_0, \varphi(t_1,\vec{x}_1)\cdots\varphi(t_n,\vec{x}_n)\Omega_0\rangle \\ &= \mathcal{W}^{(n)}(\Lambda^{-1}x_1 + a,\ldots,\Lambda^{-1}x_n + a) \ .\end{aligned}$$

The elementary *positive-energy condition* for the Hamiltonian H entails

$$0 \leqslant H \quad \text{and} \quad H\Omega_0 = 0 \ ,$$

which ensures that the Wightman functions (as functions of anti-time-ordered variables $t_{j+1} > t_j$ to purely imaginary time it_j) continue analytically to the corresponding *Schwinger functions*, *viz.*,

$$\mathcal{S}^{(n)}(t_1,\vec{x}_1,\ldots,t_n,\vec{x}_n) = \mathcal{W}^{(n)}\big((it_1,\vec{x}_1),\ldots,(it_n,\vec{x}_n)\big) \ ,$$

which play a key role in the construction of interacting quantum field theories. The Wightman functions satisfy the following identity[9],

$$\mathcal{W}^{(n)}\big((t_1\cosh\beta,\vec{x}_1+t_1\sinh\beta),\dots,(t_n\cosh\beta,\vec{x}_n+t_n\sinh\beta)\big)$$
$$=\langle\varphi(0,\vec{x}_1)\Omega_0,\mathrm{e}^{-i(t_1-t_2)\cosh\beta H_{\vec{v}}}\varphi(0,\vec{x}_2)\cdots\mathrm{e}^{-i(t_{n-1}-t_n)\cosh\beta H_{\vec{v}}}\varphi(0,\vec{x}_n)\Omega_0\rangle$$

and can be analytically continued (in the relative variables) to imaginary times. For $s_i\geqslant s_{i+1}$, where $i=1,\dots,n-1$, define the modified Schwinger functions,

$$\mathcal{S}^{(n)}_{\vec{v}}\big(t_1\cosh\beta,\vec{x}_1+t_1\sinh\beta,\dots,t_n\cosh\beta,\vec{x}_n+t_n\sinh\beta\big)$$
$$\doteq\langle\varphi(0,\vec{x}_1)\Omega_0,\mathrm{e}^{-(t_1-t_2)\cosh\beta H_{\vec{v}}}\varphi(0,\vec{x}_2)\cdots\mathrm{e}^{-(t_{n-1}-t_n)\cosh\beta H_{\vec{v}}}\varphi(0,\vec{x}_n)\Omega_0\rangle.$$

Using the Flat Tube Theorem, the possibility to vary $\vec{v}$ ensures that the Wightman functions (in difference coordinates x_j-x_{j+1}) extend by analytic continuation to the forward tube

$$\mathcal{T}^{n-1}=\mathbb{R}^{(n-1)d}-i(V_+)^{n-1}\,,\qquad V_+=\{(t,\vec{x})\in\mathbb{R}^d\mid|\vec{x}|<|t|\}\,.$$

The novelty of our approach is that the modified Schwinger functions appear as the expectation values of complex classical fields.

IX.2. **Compactified Time.** For the quantum theory reconstructed from a classical theory with periodic time, namely, $\boldsymbol{X}=S^1\times\mathbb{R}^{d-1}$ or $\boldsymbol{X}=S^1\times\mathbb{T}^{d-1}$, the spectrum condition no longer holds. In fact, the Osterwalder-Schrader reconstruction provides thermal equilibrium states, and the spectrum of both the generators of the time-evolution and of the spatial translations is symmetric around zero. Therefore, one might wonder whether quantization by reflection positivity yields a quantum field whose equations of motion are invariant under Poincaré transformations. In order to answer this question, it is instructive to compare our result with the free thermal neutral scalar Wightman field, whose expectation values are specified by the averaged two-point function,

$$\mathcal{W}^{(2)}_\beta\big((t,\alpha),(t'\alpha')\big)=\langle\alpha,\coth(\beta\mu)\mathrm{e}^{-i(t-t')\mu}\alpha'\rangle_{\mathfrak{H}_{-\frac12}}\,,\quad\alpha,\alpha'\in\mathfrak{H}_{-\frac12}(\mathbb{R}^d)\,,$$

where β denotes the inverse temperature. Although this two-point function is *not* invariant under boosts, there is no problem to consider

[9]Assuming that the $\vec{v}=(v,0,0)$ lies in the x_1-direction, the Lorentz transformation representing this boost (in $1+3$-spacetime dimensions) is given by the matrix

$$\Lambda=\begin{pmatrix}\cosh\beta&-\sinh\beta&0&0\\-\sinh\beta&\cosh\beta&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\,,\quad\cosh\beta=\frac{1}{\sqrt{1-v^2}}\,,\ \sinh\beta=\frac{v}{\sqrt{1-v^2}}\,.$$

the expectation values of Lorentz boosted quantum fields (suppressing the rescaling factor $1/\sqrt{1-\vec{v}^{\,2}}$). Since $\mu > \mu_+/(1+|\vec{v}|)$, the expression

$$\mathcal{W}_\beta^{(2)}\big((t,\vec{x}-t\vec{v})\,,\,(t',\vec{x}'-t'\vec{v})\big) = \left(\frac{1}{2\mu}\coth(\beta\mu)\mathrm{e}^{-i(t-t')\mu_+}\right)(\vec{x},\vec{x}')$$

allows an analytic continuation to imaginary times into the strip

$$\Big\{(t-t')\in\mathbb{C}\mid -\tfrac{\beta}{1+|\vec{v}|} < \Im(t-t') < 0\Big\}\,.$$

Thus, the Flat Tube Theorem ensures that $\mathcal{W}_\beta^{(2)}(t,\vec{x})$ is analyticity in the tube

$$\mathcal{T}_\beta = \mathbb{R}^d - i\left(V^+\cap(\beta\mathbf{e}_1 - V^+)\right)\ , \qquad \text{(IX.1)}$$

where $\mathbf{e}_1=(1,0,\dots,0)$ is the unit vector in the time-direction distinguished by the rest-frame.

For interacting quantum field theories, this question was addressed by Bros and Buchholz, who formulated a *relativistic KMS condition* [2, 3]. The relativistic KMS condition ensures analyticity of the two point-function in the domain (IX.1). They verified that the relativistic KMS condition holds for a large class of quantum field theories satisfying the nuclearity condition of Buchholz and Wichmann [5]. Nuclearity, however, has not been established for the models considered in Constructive Quantum Field Theory. Only recently the *relativistic KMS condition* has been proved for the $\mathscr{P}(\varphi)_2$-models using multiple reflection positivity [22].

For real $\vec{v}$, the function appearing in the corollary at the end of §VII can be expressed in terms of the two-point function,

$$\left(\tfrac{\coth(\beta\mu_+)}{2\mu}\mathrm{e}^{-i\frac{(t-t')}{1-v^2}\mu_+}\right)(\vec{x},\vec{x}') = \mathcal{W}^{(2)}_{\sqrt{1-v^2}\beta}\Big(\big(\tfrac{t+\vec{v}\cdot\vec{x}}{\sqrt{1-v^2}}\,,\,\tfrac{\vec{x}}{\sqrt{1-v^2}}\big)\,,\,\big(\tfrac{t'+\vec{v}\cdot\vec{x}'}{\sqrt{1-v^2}}\,,\,\tfrac{\vec{x}'}{\sqrt{1-v^2}}\big)\Big)\,,$$

which is analytic in the domain

$$|\Im(\vec{x}-\vec{x}')| - \beta(1-v^2) < \Im\big((t-t')+\vec{v}\cdot\vec{x}\big) < -|\Im(\vec{x}-\vec{x}')|\,,$$

and the boundary values satisfy the KMS condition for $\Im(\vec{x}-\vec{x}')=0$.

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