Twist Fields and Broken Supersymmetry*

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Abstract

A twist field on a cylindrical space-time has the defining property that translation about a spatial circle results in multiplying the field by a phase. In this paper we investigate how such multi-valued twist fields fit into the framework of constructive quantum field theory. Twisted theories have an interest in their own right; the twists also serve as infrared regulators that partially preserve the underlying symmetries of the Hamiltonian. The main focus of this paper is to investigate the extent that boson-fermion twist-field systems are compatible with the Lie symmetry and with the $N = 2$ supersymmetry that one expects in the same examples without twists. We consider free systems, and also non-linear boson-fermion interactions that arise from a holomorphic, quasi-homogeneous, polynomial superpotential. We choose the twisting angles to lie on a chosen line in twist parameter space (leaving one free twist parameter). Doing this, we can obtain Lie symmetry and half the number of supersymmetry generators that one expects in our examples without the twists. We also show that the Hamiltonians for scalar twist fields yield twisted, positive-temperature expectations with the "twist-positivity" property. This is important because it justifies the existence of a functional integral representation for twisted, positive-temperature trace functionals. We regularize these systems in a way that preserves symmetry to the maximal extent. We pursue elsewhere other aspects and applications of this method, including bounding the extent of supersymmetry breaking.

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Contents

I Introduction

The pioneering work of early non-relativistic quantum theory led to the understanding that quantum dynamics on Hilbert space is a comprehensive predictive framework for microscopic phenomena. The incorporation of special relativity and field theory into quantum theory extended the scope of perturbative calculations, and these were tested through precision measurements of spectra and magnetic moments. Beginning in the 1940's, experimental tests detected the first effects that one can ascribe to fluctuations in quantum electrodynamics, and that deviated from the predictions of equations describing a fixed number of particles. Today these experiments have evolved to yield quantitative agreement with the most precise observations and calculations achieved in physics. The success of this work, as well as the success of other less accurate, but compelling, predictions for weak and strong interactions, convince us to accept quantum field theory as correct physical arena to describe particle physics down to the Planck scale.

But the success of relativistic field theory calculations and of perturbative renormalization also led to a logical puzzle: is any physically-relevant, relativistic quantum field theory logically (mathematically) consistent? Put differently, can one give a mathematically complete example of any non-linear theory, relevant for the description of interacting particles, whose solutions incorporate relativistic covariance, positive energy, and causality? If the answer to this question is positive, can one find the properties of such examples both perturbatively and non-perturbatively? The problems that need to be solved to answer these questions include understanding renormalization divergences in perturbative calculations from a non-perturbative (or "exact") point of view. These problems also encompass understanding more sophisticated questions, such as whether a field theory may appear correct on a perturbative level, while it may have no meaning at a non-perturbative level. Related questions about quantum electrodynamics or scalar meson theory were raised early by Dyson and Landau. They recur from the point of view of the renormalization group in the work of Kadanoff and Wilson, as well as in the analysis of "asymptotic freedom" in the 1970's.

These questions remain open today for interactions in four dimensional space-time, despite the success to date of constructive quantum field theory methods. Some particle physicists have ambitious attempts to imbed quantum field theory within a theory of strings, by which they hope to combine quantum theory with general relativity, and to predict the structure of space-time. There is also the appealing attempt to integrate non-commutative geometry, as founded by Connes in the 1980's, into the picture. One would like to introduce the notion of quantization directly at the level of space-time, rather than only applying to the functions on space-time. For the time being, all these methods remain beyond the realm of full understanding.

Constructive quantum field theory (CQFT) emerged in the 1960's as a framework to show that non-linear quantum fields can be found, and that these examples actually fit within a mathematically complete description of quantum mechanics. CQFT represents a direct attack on the problem of establishing both the existence and the properties of particular examples of quantum field theory within a mathematical realm. The efforts of constructive quantum field theorists are directed not only to the justification of expected phenomena, but also to the broader exploration of physics at a fundamental level, consistent with historical precedents of mathematical integrity. The most basic questions revolve about whether examples could be found within the frameworks formulated earlier by Wightman or by Haag and Kastler. These questions can be attacked by establishing the existence of solutions to quantum field equations, thereby establishing examples of field theories satisfying the Wightman axioms (or variations on the Wightman axioms associated with a compactified space).

Fundamental progress on answering these questions led to the non-perturbative construction of field theories with non-linear interaction in two and in three dimensional space-times. Through this approach, one established the compatibility of quantum field theory with special relativity in these space-times. (See [1] for a further discussion of these and other points, as well as for references.) This work also led to establishing physical properties of these examples, including many features of their particle spectrum, the description of scattering in these examples, and the qualitative behavior of the examples as a function of the coupling constants. For example, in certain theories one can establish the existence of a second order phase transition as one varies the coupling constant. In some such cases, there are critical coupling constants for which the gap in the mass spectrum vanishes. One common constructive method follows from the discovery by Nelson, Osterwalder, and Schrader that the framework of Euclidean field theory (originally proposed by Schwinger and by Symanzik) not only can be used as the fundamental tool to investigate Minkowski field theory, but for a certain type of field theory the two approaches are precisely equivalent. Euclidean methods lead to mathematically-sound, functional-integral representations of the solutions to field theory problems, and these representations often reflect underlying symmetries of the field theories in a simple way. These techniques have been justified and realized in the lower-dimensional examples. The explicit integral representations lend themselves to the non-perturbative analysis of the examples. One has discovered expansion techniques to analyze the functional integrals in the limits as one removes an infra-red cutoff or an ultra-violet cutoff. Continued developments in the theory of renormalization and phase cell localization point to an optimistic outlook. One can envision the positive future answer to the question of the existence of an asymptotically-free, four-dimensional gauge theory on a cylindrical space-time, although the infra-red (infinite-volume) limit still seems beyond grasp.

In this paper we study twist fields on a cylindrical space-time from the point of view of constructive quantum field theory. A twist field has the defining property that translation about the spatial circle results in multiplying the field by a phase (or twist). We begin with a cylindrical space-time $\mathcal{M} \times \mathbb{R}$, the product of a spatial n-torus $\mathcal{M}$, with coordinates x, times a real-valued time $\mathbb{R}$, with coordinate t. Let $\mathcal{D}$ denote the vector space of smooth functions f on $\mathcal{M} \times \mathbb{R}$ with an appropriate topology. Let $\mathcal{H}$ denote an appropriate Hilbert space. (In the examples studied here, this Hilbert space is a bosonic or fermionic Fock space over a number of copies of $L^2(\mathcal{M})$, or a tensor product of a number of copies of these spaces.) Quantum fields $\varphi_{\rm RT}(x,t)$ are operator-valued distributions, namely linear maps from $\mathcal{D}$ to linear operators on $\mathcal{H}$. The subscript RT denotes a real-time field, and $\varphi_{\rm RT}(x,t)$ satisfies a hyperbolic partial differential equation.

Fundamental to the notion of quantum field is the assumption that the abelian group of space and time translations of $\mathcal{M}\times\mathbb{R}$ has a unitary representation on $\mathcal{H}$ generated by the self-adjoint operators P and H, called the momentum and the Hamiltonian. This translation group $e^{ixP+itH}$ also implements translation of the fields,

$$\varphi_{\rm RT}(x'-x,t'+t)=e^{ixP+itH}\varphi_{\rm RT}(x',t')e^{-ixP-itH}\;. \tag{I.1}$$

Given the constant Ω, define the *twist group* of the field $\varphi_{\rm RT}$ by

$$\varphi_{\rm RT}\to e^{i\Omega\theta}\varphi_{\rm RT}\;,\;\; \text{parameterized by }\;\theta\in\mathbb{R}\;. \tag{I.2}$$

We assume that the twist group is implemented by a unitary group $U(\theta)=e^{i\theta J}$ on $\mathcal{H}$, with the self-adjoint, infinitesimal generator J, so

$$U(\theta)\varphi_{\rm RT}U(\theta)^*=e^{i\Omega\theta}\varphi_{\rm RT}\;. \tag{I.3}$$

We assume that the group $e^{i\theta J}$ commutes with the group $e^{ixP+itH}$, so the group $e^{ixP+itH+i\theta J}$ is a three-parameter abelian group acting on $\mathcal{H}$. We also use the notation $U(g)=e^{ixP+i\theta J}$, where $g=(x,\theta)\in\mathcal{M}\times\mathbb{R}$, to denote the two-parameter abelian symmetry group of translations and twists of H.

We call the field φ a *twist field,* if the spatial translated groups and the twist group are related. In this case we assume that translation about a spatial period ℓ_j (the period of the $j^{\rm th}$-coordinate), results in a spatial twist implemented by J,

$$\varphi_{\rm RT}(x_1,x_2,\ldots,x_j+\ell_j,\ldots,x_n,t)=e^{i\chi_j}\,\varphi_{\rm RT}(x_1,\ldots,x_n,t)\;, \tag{I.4}$$

where $\chi_j=\Omega_j\theta\in[0,2\pi]$ is a fixed twisting angle. In order to achieve a regularization, we require that χ_j lie strictly between 0 and 2π. Here we generally let $\varphi_{\rm RT}$ denote a scalar field and we similarly introduce a fermionic twist field ψ with its own set of twisting angles.

In an earlier paper [2], one of us analyzed a property of bosonic fields called *twist positivity,* that leads to the existence of a countably-additive measure defining a functional integral representation for the bosonic heat kernel. Consider the bosonic field $\varphi_{\rm RT}$ acting on the bosonic Hilbert space $\mathcal{H}^b$, with a Hamiltonian H that commutes with the symmetry group $U(g)$, and with the following additional property: the Hamiltonian H has a unique ground state vector $\Omega^b_{\rm vacuum}$ and $U(g)$ is normalized so that

$$U(g)\Omega^b_{\rm vacuum}=\Omega^b_{\rm vacuum}\;. \tag{I.5}$$

Define the twisted partition function

$$\mathfrak{Z}={\rm Tr}_{\mathcal{H}^b}\left(U(g)^*e^{-\beta H}\right)={\rm Tr}_{\mathcal{H}^b}\left(e^{-ixP-i\theta J-\beta H}\right)\;. \tag{I.6}$$

We say that $\mathfrak{Z}$ has the *twist positivity* property with respect to the representation $U(g)$, if

$$\mathfrak{Z} > 0 , \quad \text{for all } g \in G , \text{ and all } \beta > 0 . \tag{I.7}$$

We show in §II that the free bosonic twist fields we introduce here are twist positive with respect to the representation $U(g) = e^{ixP+i\theta J}$ described above, and that twist fields have a Feynman-Kac representation for expectations in the twisted functional

$$\langle \cdot \rangle = \frac{\mathrm{Tr}_{\mathcal{H}^b}\left(\cdot \, e^{-ixP-i\theta J-\beta H}\right)}{\mathrm{Tr}_{\mathcal{H}^b}\left(e^{-ixP-i\theta J-\beta H}\right)} = \int \cdot \, d\mu^b_{x,\theta,\beta,\chi} , \tag{I.8}$$

where $d\mu^b_{x,\theta,\beta,\chi}$ is a countably-additive, probability measure. In the free case, this measure is Gaussian and has a covariance that is a Green's function of the form

$$C_{x,\theta,\beta,\chi} = \left(-\Delta_{x,\theta,\beta,\chi}\right)^{-1} , \tag{I.9}$$

and we find in §II.5 that $\Delta_{x,\theta,\beta,\chi}$ is a Laplacian with twisted boundary conditions depending on the parameters x, θ, β, χ. (We abstract the Gaussian twist positivity property in [3].)

In §V we introduce Dirac twist fields. In this case, we take $\mathcal{H}$ to be the tensor product of a bosonic Fock space $\mathcal{H}^b$ used in the purely bosonic examples just described, with a fermionic Fock space $\mathcal{H}^f$, so $\mathcal{H} = \mathcal{H}^b \otimes \mathcal{H}^f$. There is a similar Gaussian fermionic expectation for a free fermionic system, and in this case it is natural to also include the symmetry $\Gamma = (-I)^{N^f}$ in the expectation. Here N^f is the fermionic number operator, and the self-adjoint operator Γ has eigenvalues ± 1, and provides an additional $\mathbb{Z}_2$ symmetry. We choose J in such a way that the four operators H, P, J, and Γ mutually commute. We obtain the Green's function of Dirac operator in §V.8 of the form

$$S_{x,\theta,\beta,\chi} = \left(\partial\!\!\!/_{\,x,\theta,\beta,\chi}\right)^{-1} , \tag{I.10}$$

where the subscript again denotes twisted boundary conditions. The twisted expectation on the full Hilbert space relates to a construction in non-commutative geometry, as explained in [4]. An application to twist field theory can be found in [5] and [6].

The main question we investigate is whether a system of boson and fermion twist fields are compatible with conventional $N = 2$ supersymmetry, as characterized by the algebra

$$Q_1^2 = \widetilde{Q}_1^2 = H + P , \quad Q_2^2 = \widetilde{Q}_2^2 = H - P , \tag{I.11}$$

and the independence relations

$$Q_1Q_2 + Q_2Q_1 = Q_\alpha \widetilde{Q}_\beta + \widetilde{Q}_\beta Q_\alpha = \widetilde{Q}_1\widetilde{Q}_2 + \widetilde{Q}_2\widetilde{Q}_1 = 0 . \tag{I.12}$$

In the case of free fields, we obtain this standard supersymmetry algebra — as long as the bosonic and the fermionic twisting angles are equal, as must be the bosonic and the fermionic masses. The resulting Hamiltonian H is translation invariant under the group generated by the momentum operator P, and the Hamiltonian also possesses a one-parameter, $U(1)$ group of symmetries that we denote $U(\theta)$ and we call the *twist group.*

In §VII, we introduce interaction between bosonic and fermionic fields, mediated by a holomorphic, quasi-homogeneous, polynomial superpotential W. For an appropriate one-parameter family of twisting angles, we obtain a translation-invariant Hamiltonian which possesses a global $U(1)$-twist group symmetry $U(\theta) = e^{i\theta J}$. But in this case the twist fields are not fully compatible with the standard $N = 2$ supersymmetry algebra, and the twisting breaks supersymmetry in a regular way. Nevertheless, one can preserve one of the two components of the supercharge as an operator that is both translation-invariant and twist-invariant. This operator Q_1 (or the second copy $\widetilde{Q}_1$) is the integral of a local density, it is symmetric (we show elsewhere that it is self adjoint), and it satisfies the standard relation with the Hamiltonian and momentum operators,

$$Q_1^2 = H + P = \widetilde{Q}_1^2 \,, \tag{I.13}$$

as well as the independence relations[1]

$$[P, Q_1] = [P, \widetilde{Q}_1] = [J, Q_1] = [J, \widetilde{Q}_1] = [J, P] = Q_1\widetilde{Q}_1 + \widetilde{Q}_1 Q_1 = 0 \,. \tag{I.14}$$

We also remark on how twist fields provide a natural infra-red regularization for quantum field theory.

In §VII, we give the explicit error operators $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ that arise in those supersymmetry relations involving the second component of the supercharge. The errors in the algebra are proportional to these operators and to a twisting parameter ϕ, that is proportional to both the bosonic and to the fermionic twisting angles. The operator $\mathcal{R}$ is a fermionic number operator, independent of the superpotential W, and it commutes with both P and J. The operator $\widetilde{\mathcal{R}}$ is a Fourier mode of the superpotential, and it commutes with neither P nor J. Both operators are well behaved and amenable to the estimates of constructive quantum field theory, as we show in [7]. The error terms in the supersymmetry algebra vanish in a regular way proportional to the twisting parameter ϕ, as $\phi \to 0$. In particular,

$$Q_2^2 - (H - P) = \widetilde{Q}_2^2 - (H - P) = \phi\,\mathcal{R} \,, \tag{I.15}$$

while

$$\{Q_1, Q_2\} = \{\widetilde{Q}_1, \widetilde{Q}_2\} = \phi\left(\widetilde{\mathcal{R}} + \widetilde{\mathcal{R}}^*\right) \,, \tag{I.16}$$

and

$$\{Q_1, \widetilde{Q}_2\} = \{\widetilde{Q}_1, Q_2\} = -i\phi\left(\widetilde{\mathcal{R}} - \widetilde{\mathcal{R}}^*\right) \,. \tag{I.17}$$

This leads to the representations

$$H = \frac{1}{2}\left(Q_1^2 + Q_2^2 - \phi\mathcal{R}\right) = \frac{1}{2}\left(Q_1 + Q_2\right)^2 - \frac{1}{2}\phi\left(\mathcal{R} + \widetilde{\mathcal{R}} + \widetilde{\mathcal{R}}^*\right) \,, \tag{I.18}$$

[1]Consequently, we preserve the property that H, P, and J are mutually commuting operators. This allows us to use the twist fields in applications [7].

and

$$P = \frac{1}{2}\left(Q_1^2 - Q_2^2 + \phi\mathcal{R}\right) . \tag{I.19}$$

Finally, in §VIII we analyze these results from the point of view of superfields.

II Bosonic Twist Fields on a Torus

In this section, we consider a bosonic field φ^χ on a compact spatial manifold $\mathcal{M}$ equal to a torus. The corresponding space-time $\mathcal{M} \times \mathbb{R}$ is related to the compactified space-time $\Sigma = \mathcal{M} \times S^1$. Random fields on Σ arise when we consider certain trace functionals on the space of quantum fields.

II.1 Basic Notation

Denote the s-torus by $\mathbb{T}^s$, and let $\ell = \{\ell_1, \ell_2, \ldots, \ell_s\}$ denote its periods. The bosonic field φ^χ is a section of an n-dimensional complex vector bundle over $\mathcal{M} = \mathbb{T}^s$. In the case $n = 1$, the field is a section of a line bundle, and we quantize each component of the field φ_i as a section of a line bundle. The twist angle χ_{ij} will characterize the twist of the i^{th} component of the field under translation by one period in the j^{th}-coordinate direction. Let

$$\chi = \{\chi_{ij} : 1 \le i \le n,\ 1 \le j \le s\} , \tag{II.1}$$

denote the collection the twisting angles for all components. In case of a 2-dimensional space-time, $s = 1$ and j takes only one value, so we write

$$\chi = \{\chi_i\} , \qquad \text{where} \qquad \chi_i = \{\chi_{i1}\} . \tag{II.2}$$

Let

$$L_j = (0, 0, \ldots, \ell_j, \ldots, 0) \tag{II.3}$$

denote the s-vector with the j^{th} coordinate given by the j^{th} period of $\mathbb{T}^s$. Correspondingly, let $\ell = \{\ell_i : 1 \le i \le n\}$ denote the set of periods. By definition, each component of the twist field satisfies the relations

$$\varphi_i^\chi(x + L_j, t) = e^{i\chi_{ij}}\, \varphi_i^\chi(x, t) , \quad i = 1, 2, \ldots, n . \tag{II.4}$$

We will exclude the periodic case for any component, so we assume that

$$\chi_{ij} \notin 2\pi\mathbb{Z} , \qquad \text{for all} \qquad 1 \le i \le s,\ 1 \le j \le n . \tag{II.5}$$

Thus φ^χ must be a complex field.

We now analyze the Fourier representation of the field. The twist condition (II.4) ensures that the Fourier coefficients of the component φ_i^χ live on the union of the lattice

$$K_i^\chi = \{k \in \mathbb{R}^s : \ell_j k_j \in 2\pi\mathbb{Z} - \chi_{ij} , \quad 1 \le j \le s\} , \tag{II.6}$$

and the lattice $-K_i^\chi$. As a consequence of the assumption (II.5),

$$0 \notin K_i^\chi \ , \qquad \text{for each} \qquad 1 \leq i \leq n \ . \tag{II.7}$$

The Hilbert space $\mathcal{H}^{b,\chi}$ for a free bosonic twist field on a spatial torus $\mathcal{M} = \mathbb{T}^s$ is a Fock space (depending on the twisting angle χ). The one-particle Hilbert space for a single-component field is $l_2(K_i^\chi) \oplus l_2(-K_i^\chi)$, and in the case of a vector bundle of dimension n,

$$\mathcal{K}^\chi = \bigoplus_{i=1}^{n} \left(l_2(K_i^\chi) \oplus l_2(-K_i^\chi)\right) \ . \tag{II.8}$$

The Fock space $\mathcal{H}^{b,\chi}$ is the symmetric tensor algebra over $\mathcal{K}^\chi$,

$$\mathcal{H}^{b,\chi} = \exp_{\otimes_s} \mathcal{K}^\chi \ , \tag{II.9}$$

where $\otimes_s$ denotes the symmetric tensor product. Define two independent sets of canonical creation operators on this Fock space. For each $1 \leq i \leq n$, let

$$a_{+,i}^\chi(k)^*, \ \ k \in K_i^\chi \ , \ \text{ and } \ a_{-,i}^\chi(-k)^*, \ \ k \in K_i^\chi \ . \tag{II.10}$$

denote these operators.

The time-zero field φ^χ has components with Fourier representations

$$\varphi_i^\chi(x) = \frac{1}{\sqrt{|\mathcal{M}|}} \sum_{k \in K_i^\chi} q_i^\chi(k) e^{-ikx} \ , \tag{II.11}$$

where $|\mathcal{M}| = \ell_1 \ell_2 \cdots \ell_s$ is the volume of $\mathcal{M}$, and where for $k \in K_i^\chi$ the coordinates

$$q_i^\chi(k) = \frac{1}{(2|k|)^{1/2}} \left(a_{+,i}^\chi(k)^* + a_{-,i}^\chi(-k)\right) \tag{II.12}$$

and their adjoints $q_i^\chi(k)^*$ generate an abelian algebra. The time-zero fields (II.11) satisfy the twist relation (II.4). Similarly, the components of the conjugate field π^χ have Fourier representations

$$\pi_i^\chi(x) = \frac{1}{\sqrt{|\mathcal{M}|}} \sum_{k \in K_i^\chi} p_i^\chi(k) e^{ikx} \ , \tag{II.13}$$

where the coordinates

$$p_i^\chi(k) = -i \left(|k|/2\right)^{1/2} \left(a_{+,i}^\chi(k) - a_{-,i}^\chi(-k)^*\right) \tag{II.14}$$

and their adjoints $p_i^\chi(k)^*$ also generate an abelian algebra. Furthermore the commutation relations between the $p_i^\chi(k)$'s and the $q_i^\chi(k)$'s are canonical,

$$[p_i^\chi(k), q_{i'}^\chi(k')] = -i\delta_{ii'}\delta_{kk'} I \ , \qquad \text{and} \qquad [p_i^\chi(k), q_{i'}^\chi(k')^*] = 0 \ . \tag{II.15}$$

The conjugate fields satisfy a spatial twist relation

$$\pi_i^{\chi}(x+L_j) = e^{-i\chi_{ij}}\,\pi_i^{\chi}(x)\ . \tag{II.16}$$

We also use the number operators

$$N_{+,i}^{\chi}(k) = a_{+,i}^{\chi}(k)^* a_{+,i}^{\chi}(k)\ ,\ \text{ and }\ N_{-,i}^{\chi}(-k) = a_{-,i}^{\chi}(-k)^* a_{-,i}^{\chi}(-k)\ ,\qquad \text{for}\qquad k\in K_i^{\chi}\ . \tag{II.17}$$

In terms of these define

$$H_0^{b,\chi} \;=\; \sum_{i=1}^{n}\sum_{k\in\mathcal{K}_i^{\chi}} |k|\left(N_{+,i}^{\chi}(k) + N_{-,i}^{\chi}(-k)\right) \tag{II.18}$$

$$P^{b,\chi} \;=\; \sum_{i=1}^{n}\sum_{k\in\mathcal{K}_i^{\chi}} k\left(N_{+,i}^{\chi}(k) - N_{-,i}^{\chi}(-k)\right)\ . \tag{II.19}$$

Also define

$$J^{b,\chi} = \sum_{i=1}^{n}\sum_{k\in\mathcal{K}_i^{\chi}} \Omega_i\left(N_{+,i}^{\chi}(k) - N_{-,i}^{\chi}(-k)\right)\ , \tag{II.20}$$

where $\Omega = \{\Omega_1, \Omega_2, \ldots, \Omega_n\}$ are fixed positive constants in the interval $0 < \Omega_i \le \frac{1}{2}$. The operators $H_0^{b,\chi}$, $P^{b,\chi}$ and $J^{b,\chi}$ commute pairwise, so $P^{b,\chi}$ and $J^{b,\chi}$ generate symmetries of $H_0^{b,\chi}$. The zero-particle Fock state is annihilated by $H_0^{b,\chi}$, $P^{b,\chi}$, and $J^{b,\chi}$.

The real-time dependent field is defined by the evolution given by the Schrödinger group,

$$\varphi_{\mathrm{RT},i}^{\chi}(x,t) = e^{itH_0^{b,\chi}}\,\varphi_i^{\chi}(x)e^{-itH_0^{b,\chi}}\ , \tag{II.21}$$

namely

$$\varphi_{\mathrm{RT},i}^{\chi}(x,t) = \frac{1}{\sqrt{|\mathcal{M}|}}\sum_{k\in K_i^{\chi}}\frac{1}{(2|k|)^{1/2}}\left(e^{it|k|}a_{+,i}^{\chi}(k)^* + e^{-it|k|}a_{-,i}^{\chi}(-k)\right)e^{-ikx}\ . \tag{II.22}$$

As a consequence, the real time field satisfies the wave equation

$$\frac{\partial^2}{\partial t^2}\varphi_{\mathrm{RT}}^{\chi}(x,t) = \nabla^2\varphi_{\mathrm{RT}}^{\chi}(x,t)\ , \tag{II.23}$$

and the equal-time canonical commutation relations. It has initial data

$$\varphi_{\mathrm{RT}}^{\chi}(x,0) = \varphi^{\chi}(x)\ ,\qquad \text{and}\qquad \left(\frac{\partial\varphi_{\mathrm{RT}}^{\chi}}{\partial t}\right)(x,0) = \pi^{\chi}(x)^*\ . \tag{II.24}$$

We denote the adjoints of the time-zero fields by

$$\overline{\varphi}_i^{\chi}(x) = (\varphi_i^{\chi}(x))^*\ ,\qquad \text{and}\qquad \overline{\pi}_i^{\chi}(x) = (\pi_i^{\chi}(x))^*\ . \tag{II.25}$$

The fields satisfy the canonical commutation relations

$$[\varphi_i^\chi(x), \varphi_{i'}^\chi(y)] = [\pi_i^\chi(x), \pi_{i'}^\chi(y)] = [\varphi_i^\chi(x), \overline{\varphi}_{i'}^\chi(y)] = [\pi_i^\chi(x), \overline{\pi}_{i'}^\chi(y)] = [\pi_i^\chi(x), \overline{\varphi}_{i'}^\chi(y)] = 0 \;, \tag{II.26}$$

as well as

$$[\pi_j^\chi(x), \varphi_{j'}^\chi(y)] = -i\delta_{jj'}\delta(x-y)I \;. \tag{II.27}$$

The Dirac measure $\delta(x-y)$ on the torus $\mathbb{T}^s$ equals

$$\frac{1}{|\mathcal{M}|}\sum_{k\in K_j^{\chi=0}} e^{-ik(x-y)} \;, \tag{II.28}$$

independent of j. Hence

$$\begin{aligned} [\pi_j^\chi(x), \varphi_{j'}^\chi(y)] &= -i\delta_{jj'}\frac{1}{|\mathcal{M}|}\sum_{k\in K_j^{\chi}} e^{-ik(x-y)} I \\ &= -i\delta_{jj'}\frac{1}{|\mathcal{M}|}\sum_{k\in K_j^{\chi=0}} e^{-i\sum_{j'=1}^s \chi_{jj'}\left(x_{j'}-y_{j'}\right)/\ell_{j'}} e^{-ik(x-y)} I \\ &= -ie^{-i\sum_{j''=1}^s \chi_{jj''}\left(x_{j''}-y_{j''}\right)/\ell_{j''}}\delta_{jj'}\delta(x-y)I \\ &= -i\delta_{jj'}\delta(x-y)I \;. \end{aligned} \tag{II.29}$$

Two other unitary groups play a special role. Each component of the momentum $P^{b,\chi}$ generates a $U(1)$-translation group, so for $\sigma\in\mathbb{T}^s$,

$$\varphi^\chi(x-\sigma, t) = e^{i\sigma P^{b,\chi}}\varphi^\chi(x,t)e^{-i\sigma P^{b,\chi}} \;. \tag{II.30}$$

The other group is a $U(1)$-twist generated by the operator $J^{b,\chi}$, namely

$$U^{b,\chi}(\theta) = e^{i\theta J^{b,\chi}} \;. \tag{II.31}$$

This group acts on the field as

$$U^{b,\chi}(\theta)\varphi_j^\chi(x,t)U^{b,\chi}(\theta)^* = e^{i\Omega_j\theta}\varphi_j^\chi(x,t) \;. \tag{II.32}$$

Sometimes we use the following notation for the $(s+1)$-parameter product of these $U(1)$ groups,

$$U^{b,\chi}(\theta,\sigma) = e^{i\theta J^{b,\chi}+i\sigma P^{b,\chi}} \;. \tag{II.33}$$

We define unbounded operators on certain regular domains. We often use the domains

$$\mathcal{D}_t = \bigcup_{t'>t}\text{Range}\left(e^{-t'H_0^{b,\chi}}\right) \;, \quad t\geq 0 \;. \tag{II.34}$$

A convenient maximal domain is

$$\mathcal{D} = \mathcal{D}_0 \; . \tag{II.35}$$

A convenient minimal domain is the dense domain

$$\mathcal{D}_\infty = \bigcap_{t>0} \mathcal{D}_t \; . \tag{II.36}$$

Definition II.1.1. *We say that a bilinear form T with the domain $\mathcal{D}_t$ is $\mathcal{D}_t$-bounded, if for every $s > t$ the form*

$$e^{-sH_0^{b,\chi}} T e^{-sH_0^{b,\chi}} \tag{II.37}$$

extends to a bounded operator on $\mathcal{H}^{b,\chi}$. We say that a form is $\mathcal{D}_\infty$-bounded if it is $\mathcal{D}_t$ bounded for some $t < \infty$.

If T is a $\mathcal{D}_0$-bounded form on $\mathcal{H}^{b,\chi}$, define the $\mathcal{D}_t$-bounded form

$$T(t) = e^{-tH_0^{b,\chi}} T e^{tH_0^{b,\chi}} \; . \tag{II.38}$$

Let $T_1, T_2, \ldots, T_n$ denote $\mathcal{D}_0$-bounded forms, and let $t_1, t_2, \ldots, t_n$ denote increasing, distinct times $t_{i_1} < t_{i_2} < \cdots < t_{i_n}$. The *time-ordered product* of $T_1(t_1) \cdots T_n(t_n)$ is

$$(T_1(t_1) \cdots T_n(t_n))_+ = T_{i_1}(t_{i_1}) T_{i_2}(t_{i_2}) \cdots T_{i_n}(t_{i_n}) \; . \tag{II.39}$$

This form is $\mathcal{D}_s$-bounded, where $s = \max_j\{t_j\}$. The $\varphi^\chi(x, 0) = \varphi^\chi(x)$ is a bilinear form on $\mathcal{D}_0$. The components of the time-zero fields $\varphi_i^\chi(x)$ and $\pi_i^\chi(x)$ as well as their adjoints are $\mathcal{D}_0$-bounded.

II.2 Partition Functions

Define the twisted bosonic partition function $\mathfrak{Z}^b(\mathcal{T})$ by

$$\mathfrak{Z}^b(\mathcal{T}) = \mathrm{Tr}_{\mathcal{H}^{b,\chi}} \left(e^{-i\theta J^{b,\chi} - i\sigma P^{b,\chi} - \beta H_0^{b,\chi}} \right) \; . \tag{II.40}$$

Here $\mathcal{T}$ denotes the set of parameters that specifies the size of the space-time, the twisting angles for spatial periods and for the generator J^b, and the translation parameter σ,

$$\mathcal{T} = \{\chi, \theta\Omega, \sigma, \ell, \beta\} \; . \tag{II.41}$$

Denote by $\gamma_j(k) = \gamma_j(k, \mathcal{T})$ the function

$$\gamma_i(k) = e^{-i\theta\Omega_i - i\sigma k - \beta|k|} \; , \quad k \in K_i^\chi \; . \tag{II.42}$$

Also let

$$d_\chi = \inf_{1\le i\le n} \mathrm{dist}(0, K_i^\chi) \; , \qquad \text{and} \qquad d_\theta = \inf_{1\le i\le n} \mathrm{dist}(\Omega_i\theta, 2\pi\mathbb{Z}) \; . \tag{II.43}$$

In case all the χ_{ij} lie in the interval $(0, \pi]$, then

$$d_\chi = \inf_{1\le i\le n} \left(\sum_{j=1}^{s} (\chi_{ij}/\ell_j)^2 \right)^{1/2} \; . \tag{II.44}$$

Proposition II.2.1. *Let $\beta, d_\chi > 0$. Then*

a. *The partition function (II.40) is strictly positive and equals the convergent product*

$$\mathfrak{Z}^b(\mathcal{T}) = \prod_{i=1}^{n} \prod_{k\in K_i^\chi} \left(\frac{1}{|1-\gamma_i(k)|^2} \right) . \tag{II.45}$$

b. *For fixed $\{\beta, \ell, s\}$, there exists a constant $M_1 < \infty$ such that for all $\{\chi, \sigma, \theta\}$,*

$$0 < \mathfrak{Z}^b(\mathcal{T}) \leq \left(\frac{M_1}{d_\chi} \right)^{2n} . \tag{II.46}$$

c. *If also $\sigma = 0$, then for all $\{\chi, \theta\}$,*

$$0 < \mathfrak{Z}^b(\mathcal{T}) \leq \left(\frac{M_1}{d_\theta} \right)^{2n} . \tag{II.47}$$

d. *In each domain of uniform boundedness as specified by (b) or (c), the partition function $\mathfrak{Z}^b(\mathcal{T})$ is continuous in $\{\chi, \sigma, \theta\}$ or $\{\chi, \theta\}$ respectively.*

Remark. The positivity of the partition function is what we call *twist positivity* in [2]. Furthermore, as $\beta \to 0$, the partition function has an essential singularity, reflecting the infinite dimensionality of the system.

Proof. We establish the representation (II.45) as in TP, and so omit the details. As $\frac{1}{1-\gamma} = 1 + \frac{\gamma}{1-\gamma}$, the product (II.45) converges absolutely if

$$\sum_{i=1}^{n} \sum_{k\in K_i^\chi} \left| \frac{\gamma_i(k)}{1-\gamma_i(k)} \right| < \infty . \tag{II.48}$$

But $|\gamma_i(k)| = e^{-\beta|k|} < 1$ and $|\gamma_i(k)| \to 0$ exponentially as $|k| \to \infty$, so the product does converge absolutely.

The bound (II.46) follows from elementary lower bounds. For complex z in the unit disc, it is easy to see that

$$|1-z| \geq 1 - |z| . \tag{II.49}$$

We may write $|z| = e^{-x}$, for $0 \leq x$. Note that $1 - e^{-x} \geq \frac{1}{2}x$ for $0 \leq x \leq 1$, so we also have the bound

$$|1-z| \geq \frac{1}{2}|\ln(|z|)| , \qquad \text{in the annulus} \qquad 0 \leq \ln(|z|^{-1}) \leq 1 . \tag{II.50}$$

We now derive the estimate

$$\frac{1}{|1-\gamma_i(k)|} \le \begin{cases} \frac{2}{\beta|k|} & \text{if } \beta|k| \le 1 \\ e^{2e^{-\beta|k|}} & \text{if } \beta|k| \ge 1 \end{cases} . \tag{II.51}$$

If $\beta|k| \le 1$, we infer from (II.49)–(II.50) that

$$|1-\gamma_i(k)| \ge 1 - e^{-\beta|k|} \ge \frac{\beta|k|}{2} , \tag{II.52}$$

from which the first bound in (II.51) follows. If on the other hand $\beta|k| \ge 1$, then $|\gamma_i(k)| \le \frac{1}{2}$, and we have

$$\frac{1}{|1-\gamma_i(k)|} = \left| 1 + \frac{\gamma_i(k)}{1-\gamma_i(k)} \right| \le 1 + \frac{|\gamma_i(k)|}{1-|\gamma_i(k)|} \le 1 + 2\,|\gamma_i(k)| \le e^{2|\gamma_i(k)|} = e^{2e^{-\beta|k|}} , \tag{II.53}$$

establishing the other bound of (II.51).

By definition,

$$d_\chi \le \min\{|k|, \frac{\pi}{\ell}\} . \tag{II.54}$$

Also, either

$$|k| = d_\chi , \qquad \text{or else} \qquad \frac{\pi}{\ell} \le |k| , \tag{II.55}$$

with the first equality holding for exactly one value of $k \in K_i^\chi$. Therefore, in case $\beta|k| \le 1$, we have from (II.51),

$$\frac{1}{|1-\gamma_i(k)|^2} \le \left(\frac{2}{\beta|k|}\right)^2 \le \frac{4}{\beta^2 d_\chi^2} . \tag{II.56}$$

Such a bound also holds (with a different coefficient) for the case $\beta|k| \ge 1$. We derive this using (II.51) in the form

$$\frac{1}{|1-\gamma_i(k)|^2} \le e^{4e^{-\beta|k|}} \le e^{4/e} \le \left(\frac{3\beta d_\chi}{\beta d_\chi}\right)^2 \le \left(\frac{9\beta^2\pi^2}{\ell^2}\right) \frac{1}{\beta^2 d_\chi^2} . \tag{II.57}$$

We use either (II.56) or (II.57) in the case that $|k| = d_\chi$. For other values of k, we use the bound (II.51) directly. For these values of k, the magnitude $|k|$ is bounded away from zero, so the resulting product over $k \in K_i^\chi$ and over $1 \le i \le n$ is convergent. This completes the proof of (II.46).

We now establish (II.47). For $z = |z|e^{i\phi}$, and $|z| \le 1$, we have the lower bound

$$|1-z| \ge \left|\sin\left(\frac{\phi}{2}\right)\right| . \tag{II.58}$$

Furthermore, the definition (II.43) leads to

$$\left|\sin\left(\frac{\Omega_i\theta}{2}\right)\right| \geq \frac{d_\theta}{\pi} \ . \tag{II.59}$$

Thus

$$|1-\gamma_i(k)|^{-2} \leq \left|\sin\left(\frac{\Omega_i\theta}{2}\right)\right|^{-2} \leq \left(\frac{\pi}{d_\theta}\right)^2 \ . \tag{II.60}$$

Use this bound in order to estimate $|1-\gamma_i(k)|^{-2}$ for the modes for which $|k| = d_\chi$. (These modes may have $|k|$ arbitrarily small.) Estimate the remaining modes, for which $|k| \geq \pi/\ell$, using the bound (II.51) in the same fashion as in proving (II.46). This completes the proof of (II.47).

Finally, the claimed continuity of (c) follows from a direct estimate of the difference of the representation (II.45) when evaluated at two distinct values of the parameters. For example, denoting a changed parameter by a prime,

$$\left|\left(\frac{1}{1-\gamma_i(k)}\right) - \left(\frac{1}{1-\gamma_i'(k)}\right)\right| = \left|(\gamma_i'(k)-\gamma_i(k))\left(\frac{1}{1-\gamma_i(k)}\frac{1}{1-\gamma_i'(k)}\right)\right| \ . \tag{II.61}$$

Let us vary the angle θ. Then

$$|(\gamma_i'(k)-\gamma_i(k))| = 2\left|\sin\left(\frac{(\theta-\theta')\,\Omega_i}{2}\right)\right| \ . \tag{II.62}$$

This difference is $O(|\theta-\theta'|)$. The convergence of the sum over the differences in these factors does not influence the estimate of convergence of the product, and the continuity in θ follows. The proof of continuity in the other parameters is similar, and we omit further details.

II.3 The Gaussian Expectation and its Pair Correlation Matrix

Define the normalized expectation

$$\langle\ \cdot\ \rangle_{\mathcal{T}} = \frac{\mathrm{Tr}_{\mathcal{H}^{b,\chi}}\left(\ \cdot\ e^{-i\theta J^{b,\chi}-i\sigma P^{b,\chi}-\beta H_0^{b,\chi}}\right)}{\mathrm{Tr}_{\mathcal{H}^{b,\chi}}\left(e^{-i\theta J^{b,\chi}-i\sigma P^{b,\chi}-\beta H_0^{b,\chi}}\right)} \ . \tag{II.63}$$

In this subsection we define the pair correlation function, and we establish the Gaussian nature of the expectation (II.63). Since the proof of each result closely follows the proofs of Propositions VI.3, II.3, and VI.2 of [2], we only state the results.

Introduce the *imaginary time field* $\varphi^\chi(x,t)$, that is related to the real time field (II.21) by

$$\varphi^\chi(x,t) = {\varphi^\chi}_{\mathrm{RT}}(x,it) = e^{-tH_0^{b,\chi}}\varphi(x)e^{tH_0^{b,\chi}} \ . \tag{II.64}$$

The field $\varphi^\chi(x,t)$ is $\mathcal{D}_t$-bounded. Also introduce the $\mathcal{D}_t$-bounded field

$$\overline{\varphi}^\chi(x,t) = e^{-tH_0^{b,\chi}}\overline{\varphi}^\chi(x)e^{tH_0^{b,\chi}} . \tag{II.65}$$

Let $\varphi^\#$ denote a component of either φ^χ or $\overline{\varphi}^\chi$. In an identity, we need to make the same choice of # applied to a given factor on both sides of an identity.

Definition II.3.1. *Let $t_1,\ldots,t_n$ be distinct times with $t_{i_1} < t_{i_2} < \cdots < t_{i_n}$. The time ordered product of $\varphi^\#(x_1,t_i)$, $\varphi^\#(x_2,t_2),\ldots,\varphi^\#(x_n,t_n)$ is*

$$\Big(\varphi^\#(x_1,t_1)\varphi^\#(x_2,t_2)\cdots\varphi^\#(x_n,t_n)\Big)_+ = \varphi^\#(x_{i_1},t_{i_1})\varphi^\#(x_{i_2},t_{i_2})\cdots\varphi^\#(x_{i_n},t_{i_n}) . \tag{II.66}$$

Definition II.3.2. *The pair correlation matrix $C_{\mathcal{T}}(x-y,t-s)_{ij}$ of the field φ^χ is the expectation*

$$C_{\mathcal{T}}(x-y,t-s)_{ij} = \Big\langle\Big(\overline{\varphi}_i^\chi(x,t)\varphi_j^\chi(y,s)\Big)_+\Big\rangle_{\mathcal{T}} , \tag{II.67}$$

defined for $0\le t,s\le\beta$.

Proposition II.3.3. *With the notation above,*

$$C_{\mathcal{T}}(x-y+L_j,t-s)_{ii'} = \delta_{ii'}e^{-i\chi_{ij}}C_{\mathcal{T}}(x-y,t-s) , \quad j=1,2,\ldots,s , \tag{II.68}$$

and

$$C_{\mathcal{T}}(x-y,t-s+\beta)_{ii'} = \delta_{ii'}e^{-i\Omega_i\theta}C_{\mathcal{T}}(x-y-\sigma,t-s) . \tag{II.69}$$

Also

$$\langle\varphi_i^\chi(x,t)\rangle_{\mathcal{T}} = \langle\overline{\varphi}_i^\chi(x,t)\rangle_{\mathcal{T}} = \Big\langle\Big(\varphi_i^\chi(x,t)\varphi_j^\chi(y,s)\Big)_+\Big\rangle_{\mathcal{T}} = \Big\langle\Big(\overline{\varphi}_i^\chi(x,t)\overline{\varphi}_j^\chi(y,s)\Big)_+\Big\rangle_{\mathcal{T}} = 0 . \tag{II.70}$$

Since the time ordered product of fields is symmetric,

$$\Big\langle\Big(\varphi_i^\chi(x,t)\overline{\varphi}_j^\chi(y,s)\Big)_+\Big\rangle_{\mathcal{T}} = \Big\langle\Big(\overline{\varphi}_j^\chi(y,s)\varphi_i^\chi(x,t)\Big)_+\Big\rangle_{\mathcal{T}} , \tag{II.71}$$

and the pair correlation matrix defined above is hermitian, the other non-zero pair correlation matrix equals

$$\Big\langle\Big(\varphi_i^\chi(x,t)\overline{\varphi}_j^\chi(y,s)\Big)_+\Big\rangle_{\mathcal{T}} = \overline{C_{\mathcal{T}}(x-y,t-s)_{ij}} . \tag{II.72}$$

Proposition II.3.4. *The functional (II.63) is Gaussian, namely*

$$\Big\langle\Big(\varphi_{j_1}^\#(x_1,t_1)\cdots\varphi_{j_n}^\#(x_n,t_n)\Big)_+\Big\rangle_{\mathcal{T}} = \sum_{\text{pairings}}\Big\langle\Big(\varphi_{j_{i_1}}^\#(x_{i_1},t_{i_1})\varphi_{j_{i_2}}^\#(x_{i_2},t_{i_2})\Big)_+\Big\rangle_{\mathcal{T}}\cdots\Big\langle\Big(\varphi_{j_{i_{n-1}}}^\#(x_{i_{n-1}},t_{i_{n-1}})\varphi_{j_{i_n}}^\#(x_{i_n},t_{i_n})\Big)_+\Big\rangle_{\mathcal{T}} . \tag{II.73}$$

Here the functional vanishes for odd n. If n is even, the sum runs over the $(n-1)!! = \frac{n!}{2^{n/2}(n/2)!}$ pairings $\{(i_1, i_2), (i_3, i_4), \ldots, (i_{n-1}, i_n)\}$ of the n indices $\{1, \ldots, n\}$.

II.4 Evaluating the Pair Correlation Matrix

Introduce the $(s+1)$-torus $\Sigma = \mathbb{T}^s \times S^1$, where S^1 denotes the circle with period β. Let $L = \{\ell_1, \ell_2, \ldots, \ell_s, \beta\}$ denote the set of periods of Σ, and let $|\Sigma| = \ell_1 \ell_2 \cdots \ell_s \beta$ denote the volume. We introduce a vector bundle $\mathcal{S}_{\mathcal{T}}(\Sigma)$ of C^∞, multi-valued functions on Σ with the i^{th} component satisfying

$$f_i(x+L_j, t) = e^{-i\chi_{ij}} f_i(x,t) \ , \quad \text{for all } 1 \leq j \leq s, \ \text{ and } \qquad f_i(x, t+\beta) = e^{-i\Omega_i\theta} f_i(x+\sigma, t) \ . \quad \text{(II.74)}$$

Here L_j denotes the period displacements (II.3). Functions satisfying (II.74) have a Fourier representation

$$f_i(x,t) = \frac{1}{\sqrt{|\Sigma|}} \sum_{(k,E)\in\hat{\Sigma}} \hat{f}(k,E) e^{ikx+iEt-i\Omega_i\theta t/\beta+i(k\cdot\sigma)t/\beta-i\sum_{j=1}^{s}\chi_{ij}x_j/\ell_j} \ . \quad \text{(II.75)}$$

The lattice $\hat{\Sigma}$ denotes

$$\hat{\Sigma} = \{(k,E) : \ell_j k_j \in 2\pi\mathbb{Z} \ , \quad j = 1, 2, \ldots, s, \ \text{ and } \quad \beta E \in 2\pi\mathbb{Z}\} \ . \quad \text{(II.76)}$$

Smoothness of the functions $f(x,t) \in \mathcal{S}_{\mathcal{T}}(\Sigma)$ entails that the coefficients $\hat{f}(k,E)$ decrease rapidly as a function of k and E. The space $\mathcal{S}_{\mathcal{T}}(\Sigma)$ is a dense subspace of $\oplus_{i=1}^n L^2(\Sigma)$, with the inner product

$$\langle f, g \rangle = \sum_{i=1}^n \int_\Sigma \overline{f_i}(x,t) g_i(x,t) d^s x dt \ . \quad \text{(II.77)}$$

Define the operators $D_j = -i\frac{\partial}{\partial x_j}$ and $D_t = -i\frac{\partial}{\partial t}$ with the domain $\mathcal{S}_{\mathcal{T}}(\Sigma) \subset \oplus_{i=1}^n L^2(\Sigma)$. Designate the closures of these operators by $D_j^{\mathcal{T}}$ and $D_t^{\mathcal{T}}$. The superscript $\mathcal{T}$ designates the twisting parameters (II.41) for functions in the original domain of definition of the operators.

Proposition II.4.1. *The operators $D_j^{\mathcal{T}}$ and $D_t^{\mathcal{T}}$ are self-adjoint.*

Proof. We see that D_j and D_t are hermitian operators on the domain $\mathcal{S}_{\mathcal{T}}(\Sigma)$. For example, we claim that

$$\begin{aligned} \langle f, D_j g \rangle &= \sum_{i=1}^n \int_\Sigma \overline{f_i} D_j g_i \, d^s x dt \\ &= \sum_{i=1}^n \int_\Sigma \overline{D_j f_i} g_i \, d^s x dt + \sum_{i=1}^n \int_\Sigma D_j(\overline{f_i} g_i) \, d^s x dt \\ &= \sum_{i=1}^n \int_\Sigma \overline{D_j f_i} g_i \, d^s x dt = \langle D_j f, g \rangle \ . \end{aligned} \quad \text{(II.78)}$$

To justify (II.78), we verify that

$$\sum_{i=1}^{n} \int_{\Sigma} D_j(\overline{f_i} g_i)\, d^s x dt = 0 \ , \tag{II.79}$$

The boundary condition (II.74) ensures

$$\overline{f_i(x,t)} g_i(x,t) \mid_{x_j=0}^{x_j=\ell_j} = 0 \ . \tag{II.80}$$

Hence

$$\sum_{i=1}^{n} \int_{\Sigma} D_j(\overline{f_i} g_i)\, d^s x dt = \sum_{i=1}^{n} \int \overline{f_i(x,t)} g_i(x,t) dx_1 \cdots \hat{dx_j} \cdots dx_s dt \mid_{x_j=0}^{x_j=\ell_j} = 0 \ , \tag{II.81}$$

where $\hat{dx_j}$ denotes the lack of integration over the j^{th} coordinate, completing the proof of (II.78).

In a similar fashion, we infer that

$$\int_{\mathcal{M}} \overline{f(x,t)} g(x,t)\, dx \mid_{t=0}^{t=\beta} = 0 \ , \tag{II.82}$$

as a consequense of (II.74) and the translation invariance of the inner product on $L^2(\mathcal{M})$. Therefore we may repeat the above argument to demonstrate that D_t is hermitian on the domain $\mathcal{S}_{\mathcal{T}}(\Sigma)$.

For fixed $\mathcal{T}$ and Σ, define the following functions in the i^{th} component of $\mathcal{S}_{\mathcal{T}}(\Sigma)$,

$$e_{i,k,E}(x,t) = \frac{1}{\sqrt{|\Sigma|}} e^{ikx + iEt - i\Omega_i \theta t/\beta - i(k\cdot\sigma)t/\beta - i\sum_{j=1}^{s} \chi_{ij} x_j/\ell_j} \ , \tag{II.83}$$

where $(k,E) \in \hat{\Sigma}$ and $1 \le i \le n$. These functions form an orthnormal basis for $\oplus_{i=1}^{n} L^2(\Sigma)$, since they differ from the standard Fourier basis by a unitary transformation. Furthermore they are simultaneous eigenfunctions of $D_1, \ldots, D_s$, and D_t, with eigenvalues $k_j - \chi_{ij}/\ell_j$ in the case of D_j, and $E - (\Omega_i \theta + k\cdot\sigma)/\beta$ in the case of D_t. Thus each operator D_j or D_t in question has a basis of eigenfunctions, and therefore has self adjoint closure, completing the proof of Proposition II.4.1.

We define the positive Laplacian operator $\Delta_{\mathcal{T}}$ on $\oplus_{i=1}^{n} L^2(\Sigma)$. This is a diagonal matrix on the n copies of $L^2(\Sigma)$, so that on the i^{th}-copy of $L^2(\Sigma)$ it acts as

$$\Delta_{\mathcal{T}_i} = (D_t^{\mathcal{T}_i})^2 + \sum_{j=1}^{s} (D_j^{\mathcal{T}_i})^2 \ , \tag{II.84}$$

where $\mathcal{T}_i$‘denotes the twist conditions for the i^{th}-component of the field. Since this Laplacian leaves the domain $\mathcal{S}_{\mathcal{T}}(\Sigma)$ invariant, the Laplacian is essentially self-adjoint on $\mathcal{S}_{\mathcal{T}}(\Sigma)$. The spectrum of $\Delta_{\mathcal{T}}$ is discrete and does not include 0. Hence $\Delta_{\mathcal{T}}$ is invertible.

Proposition II.4.2. *The pair correlation matrix* $C_{\mathcal{T}}(x-y,t-s)_{ij}$ *defined in (II.67) equals the matrix of integral kernels of the operator* $\Delta_{\mathcal{T}}^{-1}$ *acting on* $\oplus_{i=1}^{n} L^2(\Sigma)$. *The Fourier representation of* $C_{\mathcal{T}}(x-y,t-s)_{ij}$ *is*

$$C_{\mathcal{T}}(x-y,t-s)_{ij} = \delta_{ij} \frac{1}{|\Sigma|} \sum_{(k,E)\in\hat{\Sigma}_{\mathcal{T}_i}} \frac{1}{E^2+k^2} e^{ik(x-x')+iE(t-t')} , \tag{II.85}$$

where

$$\hat{\Sigma}_{\mathcal{T}_i} = \Big\{ (k,E) : (k_j + \chi_{ij}/\ell_j,\ E + (\Omega_i\theta - (k\cdot\sigma))/\beta) \in \hat{\Sigma} \Big\} . \tag{II.86}$$

Proof. The proof follows the proof of Propositions III.1 and (VI.4) of [2]. The main difference is that we set the mass m of [2] to zero. We may do this, as the twist χ ensures that the null space of $\Delta_{\mathcal{T}}$ is empty. We omit the details.

II.5 Random Fields and the Feynman-Kac Identity

Recall that $\mathcal{S}_{\mathcal{T}}(\Sigma)$ denotes the space of C^∞, but multi-valued functions on Σ, that satisfy the relations (II.74). Endow $\mathcal{S}_{\mathcal{T}}(\Sigma)$ with the standard Frêchet topology determined by the countable norms,

$$|||f|||_i = ||D_t^{i_0} D_1^{i_1} \cdots D_n^{i_n} f||_{L^2(\Sigma)} . \tag{II.87}$$

The space of *random fields* $\Phi^{\mathcal{T}}(x,t)$ is the space of generalized functions $\mathcal{S}'_{\mathcal{T}}(\Sigma)$ topologically dual to $\mathcal{S}_{\mathcal{T}}(\Sigma)$. The pairing between $\mathcal{S}'_{\mathcal{T}}(\Sigma)$ and $\mathcal{S}_{\mathcal{T}}(\Sigma)$ has the form

$$\Phi^{\mathcal{T}}(f) = \sum_{i=1}^{n} \int_\Sigma \Phi_i^{\mathcal{T}}(x,t) f_i(x,t)\, d^s x\, dt . \tag{II.88}$$

Since this pairing is real, the adjoint operator $C_{\mathcal{T}}^{+}$ to the pair correlation operator $C_{\mathcal{T}}$ acts on the random fields according to the definition

$$\left(C_{\mathcal{T}}^{+} \Phi^{\mathcal{T}} \right)(f) = \Phi^{\mathcal{T}}(C_{\mathcal{T}} f) . \tag{II.89}$$

Note that $\mathcal{S}'_{\mathcal{T}}(\Sigma)$ contains a subspace of smooth functions, namely functions $\Phi^{\mathcal{T}} \in \mathcal{S}_{\mathcal{T}^+}(\Sigma)$, with the dual parameters given by

$$\mathcal{T}^{+} = \{-\chi, -\theta\Omega, \sigma, \ell, \beta\} . \tag{II.90}$$

Correspondingly as operators on $L^2(\Sigma)$,

$$C_{\mathcal{T}}^{+} = C_{\mathcal{T}^{+}} , \tag{II.91}$$

and we have twist relations

$$\Phi_i^{\mathcal{T}}(x+L_j,t) = e^{i\chi_{ij}} \Phi_i^{\mathcal{T}}(x,t) , \tag{II.92}$$

and

$$\Phi_i^{\mathcal{T}}(x, t+\beta) = e^{i\Omega_i\theta}\Phi_i^{\mathcal{T}}(x-\sigma, t)\ . \tag{II.93}$$

We also have

$$C_{\mathcal{T}}^{+}(x+L_j, t)_{ik} = \delta_{ik}e^{i\chi_{ij}}C_{\mathcal{T}}^{+}(x,t)_{ik}\ , \tag{II.94}$$

and

$$C_{\mathcal{T}}^{+}(x, t+\beta)_{ik} = \delta_{ik}e^{i\Omega_i\theta}C_{\mathcal{T}}^{+}(x-\sigma,t)_{ik}\ . \tag{II.95}$$

Let $d\mu_{\mathcal{T}}$ denote the Gaussian probability measure on $\mathcal{S}'_{\mathcal{T}}(\Sigma)$ with mean zero, and with covariance matrix equal to $(C_{\mathcal{T}})_{ij}$. In more detail, let

$$\Phi^{\mathcal{T}} = \left\{\Phi_1^{\mathcal{T}}, \Phi_2^{\mathcal{T}}, \ldots, \Phi_n^{\mathcal{T}}\right\} \tag{II.96}$$

and

$$\int \Phi_i^{\mathcal{T}} d\mu_{\mathcal{T}} = 0\ , \tag{II.97}$$

and

$$\int \overline{\Phi}_i^{\mathcal{T}}(x,t)\Phi_j^{\mathcal{T}}(y,s)d\mu_{\mathcal{T}} = C_{\mathcal{T}}(x-y, t-s)_{ij} = \left\langle \left(\overline{\varphi}_i^{\chi}(x,t)\varphi^{\chi}{}_j(y,s)\right)_+\right\rangle_{\mathcal{T}}\ . \tag{II.98}$$

Proposition II.5.1. *The Feynman-Kac identity holds, namely*

$$\left\langle \left(\varphi_{i_1}^{\#}(x_1,t_1)\varphi_{i_2}^{\#}(x_2,t_2)\cdots\varphi_{i_n}^{\#}(x_n,t_n)\right)_+\right\rangle_{\mathcal{T}} = \int \Phi_{i_1}^{\mathcal{T}}(x_1,t_1)^{\#}\Phi_{i_2}^{\mathcal{T}}(x_2,t_2)^{\#}\cdots\Phi_{i_n}^{\mathcal{T}}(x_n,t_n)^{\#}d\mu_{\mathcal{T}}\ . \tag{II.99}$$

Proof. The functional on the left side of (II.99) is Gaussian by Proposition II.3.4. The functional on the right side is Gaussian by definition. The first and second moments coincide by Proposition II.4.2, and the definition of $d\mu_{\mathcal{T}}$. Therefore the functionals agree.

III Infra-red Regularization

Twist fields defined on a compact manifold provide an infra-red regularization for quantum fields. This can be traced to the lack of a constant Fourier mode in the representation (II.75), and as a consequence the corresponding free fields are infra-red regular. This regularity carries over to interacting (nonlinear) quantum fields with certain polynomial non-linearities in the energy density. In this section we compare three mechanisms for regularizing fields at low-momentum:

(i) regularization by using a twist field,

(ii) regularization by introducing a mass $m > 0$, and

(iii) regularization using the classical string theory method.

We compare these methods for bosonic fields on a circle, and we prove that when used with certain stable interactions they give the same expectations after removal of the regularization.

One reason for having alternative regularization schemes is the possibility that different regularization procedures may be compatible with different symmetries. The existence of a Lie symmetry or sypersymmetry in the regularized problem may be essential. For example, an equivariant index requires the exact Lie symmetry group. In [8] we studied a symmetry that is destroyed by introducing a mass, and in using such a regularization we required a detailed argument to recover the desired invariant as we remove the regularization. Twist fields provide an alternative regularization that both preserves the symmetry and supersymmetry for the examples in [8]. The price one pays is that the regularized theories with different values of χ live on different Hilbert spaces $\mathcal{H}^{b,\chi}$, and the Feynman-Kac representations are integrals over spaces of generalized functions that depend on the twist angle χ. This can complicate identifying the limits one obtains as $\chi \to 0$.

In order to take into account theories that live on a family of Hilbert spaces, we consider the field theories as defined by sequences of expectations. By definition, two limits will agree if they have the same expectations of fields. From these expectations we reconstruct the Hilbert space, fields, Hamiltonian, and symmetries using the Wightman, GNS, Osterwalder-Schrader, or other similar reconstruction theorems.

III.1 Hamiltonians and Regularizations

In this section we study the free-field Hamiltonians H_0^b associated with the appropriate fields. We also consider Hamiltonians of the form

$$H^b = H_0^b + V \ , \tag{III.1}$$

that are (nonlinear) perturbations of a free Hamiltonian. In each of the three cases, the free-field problem will have an energy H_0^b, a momentum P^b and a symmetry generator J^b. We choose a perturbed Hamiltonian with the same P^b and we choose the weights Ω (that occur in J^b) so that J^b generates a symmetry of H^b. In other words, with

$$U(\sigma,\theta) = e^{i\theta J^b + i\sigma P^b} \ , \tag{III.2}$$

we require that for all real θ, σ, and for $\beta \geq 0$,

$$U(\sigma,\theta)e^{-\beta H^b} = e^{-\beta H^b}U(\sigma,\theta) \ . \tag{III.3}$$

We generate the perturbation V in (III.1) from a polynomial W such that

(QH) The polynomial $W(z) : \mathbb{C}^n \to \mathbb{C}$ is a holomorphic and quasi-homogeneous, and

(EL) The polynomial W satisfies certain elliptic bounds.

In more detail, let W_j denote the j^{th}-component of the gradient of W,

$$W_j(z) = \frac{\partial W(z)}{\partial z_j} \ . \tag{III.4}$$

The polynomial $W(z)$ is quasi-homogeneous if there are a set of rational numbers $\Omega = \{\Omega_j\}$ for which

$$W(z) = \sum_{j=1}^{n} \Omega_j z_j W_j(z) \ . \tag{III.5}$$

A homogeneous polynomial $W(z)$ has equal weights. Here we assume that the rational weights satisfy

$$\Omega = \left\{ \Omega_j : 0 < \Omega_j \leq \frac{1}{2} \ , \quad 1 \leq j \leq n \right\} \ . \tag{III.6}$$

In particular, this excludes constant or linear terms from $W(z)$. We say that the set of holomorphic, quasi-homogeneous polynomials with a given set of weights belong to a *class* of holomorphic, quasi-homogeneous polynomials characterized by Ω.

The relation (III.5) is the infinitesimal form of a $U(1)$ symmetry group acting on $W(z)$, parameterized by a real angle θ. The group acts on coordinates in $\mathbb{C}^n$ by $z_j \to e^{i\Omega_j \theta} z_j$, and it acts on the polynomial W by

$$W\left(e^{i\Omega_j\theta} z_j\right) = e^{i\theta} W\left(z\right) \ . \tag{III.7}$$

As a consequence of (III.7), the real polynomial

$$V(z) = |\mathrm{grad}\, W(z)|^2 = \sum_{j=1}^{n} |W_j(z)|^2 \tag{III.8}$$

is an invariant polynomial. In other words,

$$V\left(e^{i\Omega_i\theta} z_i\right) = V(z) \ . \tag{III.9}$$

In the following, we begin by taking the interaction V in (III.1) to have the form

$$V = \int_{S^1} : V\left(\varphi_j^{\mathrm{cutoff}}(x)\right) : dx \ , \tag{III.10}$$

where $\varphi^{\mathrm{cutoff}}$ denotes one of the three types of regularized fields that we discuss. In later sections, we study a bilocal approximation to this interaction.

We now consider the invariance of an interaction V under translations and twists. Let us choose the coefficients Ω_j in the definition (II.20) of J^b to be the weights (III.6) that characterize the quasi-homogeneous class of the polynomial W. Then

$$e^{i\theta J^b} V e^{-i\theta J^b} = V \ . \tag{III.11}$$

In other words, V is invariant under the action of the $U(1)$ twist group $e^{i\theta J^b}$.

It is also important that our Hamiltonian also be translation invariant. This will lead to another assumption on our fields, namely to a restriction on the twist angles. Translation invariance of V is a consequence of periodicity of the energy density $V(\varphi^{\text{cutoff}}(x))$. Our regularized fields have the property

$$\varphi_i^{\text{cutoff}}(x+L_j) = \begin{cases} e^{i\chi_{ij}}\varphi_i^{\text{cutoff}}(x)\ , & \text{case (i)} \\ \varphi_i^{\text{cutoff}}(x)\ , & \text{cases (ii) and (iii),} \end{cases} \tag{III.12}$$

where L_j are the spatial period (II.3). We are therefore led to the restriction on the twisting angles, reducing the freedom of the twists to one real parameter ϕ, and we pose this as the following hypothesis.

(TA) Choose the twist angles χ_{ij} to satisfy

$$\chi_{ij} = \Omega_i \phi\ . \tag{III.13}$$

In particular, χ_{ij} is independent of j, and $\chi_{ij}/\chi_{i'j} = \Omega_i/\Omega_{i'}$. With this choice, the potential function $V(\varphi^{\text{cutoff}}(x))$ is periodic in each coordinate direction. For all three regularizations,

$$V\left(\varphi^{\text{cutoff}}(x+L_j)\right) = V\left(\varphi^{\text{cutoff}}(x)\right)\ , \tag{III.14}$$

for $1 \le j \le s$. As a consequence,

$$U(\sigma,\theta)V = VU(\sigma,\theta)\ . \tag{III.15}$$

By construction the free Hamiltonians H_0^b are also invariant under $U(\sigma,\theta)$, so

$$U(\sigma,\theta)H^b = H^b U(\sigma,\theta)\ . \tag{III.16}$$

So far we have discussed properties (a) and (b) of the polynomial W. Property (c) is the analytic information we require in order to establish essential self-adjointness of the sum (III.1), and to lift the symmetry (III.16) of the Hamiltonian to a symmetry of the heat kernels (III.3). The requirement that W is elliptic means that $|\operatorname{grad} W|$ grows at infinity. First there exist constants M_1, $M_2 < \infty$ such that

$$|z| \le M_1 |\operatorname{grad} W(z)| + M_2\ . \tag{III.17}$$

Secondly, for any monomial derivative $D^j = \left(\frac{\partial}{\partial z_1}\right)^{j_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{j_n}$ of total degree $|j| = j_1 + \cdots + j_n \ge 2$, and for any given $\epsilon > 0$, there exists $M_3 < \infty$ such that

$$|D^j W| \le \epsilon |\operatorname{grad} W| + M_3\ . \tag{III.18}$$

The bound (III.18) allows us to estimate the normal ordering terms in (III.10). We take the domain of definition of H^b to be

$$\mathcal{D} = \bigcap_{\beta>0} \text{Range}\left(e^{-\beta H_0^b}\right)\ , \tag{III.19}$$

that is consistent with the previous definition (II.36).

III.2 Regularized Fields

We specify the regularized fields we use in this section.

III.2.1 Twist Fields φ^χ

The twist field φ^χ acting on the Hilbert space $\mathcal{H}^{b,\chi}$ has been introduced in §II, and we do not discuss it further here.

III.2.2 Massive Fields φ^m

The second type of field is the massive field, and this may be introduced with or without a twist. For simplicity we take zero twist and denote the massive field by φ^m. This field lives on the Hilbert space $\mathcal{H}^b = \mathcal{H}^{b,\chi=0}$. The time zero field φ^m and its conjugate π^m as well as the operators H_0^b, P^b, and J^b all have expressions which are minor modifications of those in §II.1. In particular, we use the canonical variables $a_{\pm,i}(k) = a_{\pm,i}^{\chi_i=0}(k)$ and their adjoints. The relativistic energy expression

$$\mu_m(k) = (k^2 + m^2)^{1/2} \tag{III.20}$$

occurs in many formulas. For example, in place of (II.11) and (II.13), we have

$$\varphi_i^m(x) = \frac{1}{\sqrt{|\mathcal{M}|}} \sum_{k\in K} q_i^m(k) e^{-ikx} , \qquad \text{and} \qquad \pi_i^m(x) = \frac{1}{\sqrt{|\mathcal{M}|}} \sum_{k\in K} p_i^m(k) e^{ikx} . \tag{III.21}$$

Here

$$K = K_i^{\chi=0} = \{ \ k \in \mathbb{R}^s : \ell_j k_j \in 2\pi\mathbb{Z}, \quad 1 \le j \le s \ \} , \tag{III.22}$$

is independent of i, and

$$q_i^m(k) = \frac{1}{\sqrt{2\mu_m(k)}} \left(a_{+,i}(k)^* + a_{-,i}(-k)\right) , \qquad \text{and} \qquad p_i^m(k) = -i\sqrt{\frac{\mu_m(k)}{2}} \left(a_{+,i}^m(k) - a_{-,i}^m(-k)^*\right) . \tag{III.23}$$

We denote the number operators for these modes by

$$N_{\pm,i}^a(k) = a_{\pm,i}(k)^* a_{\pm,i}(k) \ , \ \text{ where } \ k \in K = -K \ . \tag{III.24}$$

The expression of the free Hamiltonian as the integral of a density takes the form

$$\begin{aligned} H_0^{b,m} &= \sum_{i=1}^n \int_{\mathbb{T}^s} : \overline{\pi}_i^m \pi_i^m + \sum_{j=1}^s \partial_j \overline{\varphi}_i^m \partial_j \varphi_i^m + m^2 \overline{\varphi}_i^m \varphi_i^m : d^s x \\ &= \sum_{i=1}^n \sum_{k\in K} \mu(k) \left(N_{+,i}^a(k) + N_{-,i}^a(-k) \right) \end{aligned} \tag{III.25}$$

The real-time field φ^m_{RT} is

$$\varphi^m_{\mathrm{RT},i}(x,t) = e^{itH_0^{b,m}}\varphi^m_i(x)e^{-itH_0^{b,m}} = \frac{1}{\sqrt{\mathcal{M}}}\sum_{k\in K} q^m_i(k,t)e^{-ikx} \ , \tag{III.26}$$

where

$$q^m_i(k,t) = \frac{1}{\sqrt{2\mu_m(k)}}\left(a_{+,i}(k)^* e^{i|k|t} + a_{-,i}(-k)e^{-i|k|t}\right) \ . \tag{III.27}$$

The momentum operator and twist generator are

$$P^{b,m} = \sum_{i=1}^{n}\sum_{k\in K} k\left(N^a_{+,i}(k) - N^a_{-,i}(-k)\right) \ , \tag{III.28}$$

and

$$J^{b,m} = \sum_{i=1}^{n}\sum_{k\in K} \Omega_i\left(N^a_{+,i}(k) - N^a_{-,i}(-k)\right) \ . \tag{III.29}$$

III.2.3 String Fields φ^{str}

The third infra-red field we call the "classical string scheme" and denote the field by $\varphi^{\mathrm{str}}(x)$. In this case the field also lives on the Hilbert space $\mathcal{H}^b$. In this case, the field is identical to the $m=0$ limit of $\varphi^m_i(x,t)$, except in the constant Fourier modes; in fact the constant modes of φ^m_i have no $m\to 0$ limit. Instead, define the time-zero field as

$$\varphi^{\mathrm{str}}_i(x) = \frac{1}{\sqrt{|\mathcal{M}|}}\sum_{k\in K} q^{\mathrm{str}}_i(k)e^{-ikx} \ , \tag{III.30}$$

where

$$q^{\mathrm{str}}_i(k) = \begin{cases} \frac{1}{\sqrt{2}}\left(a_{+,j}(k)^* + a_{-,j}(-k)\right) \ , & \text{for } k=0 \\ \frac{1}{\sqrt{2|k|}}\left(a_{+,j}(k)^* + a_{-,j}(-k)\right) \ , & \text{for } k\in K, \text{ and } k\neq 0. \end{cases} \tag{III.31}$$

Similarly, define the conjugate time-zero string field as

$$\pi^{\mathrm{str}}_i(x) = \frac{1}{\sqrt{|\mathcal{M}|}}\sum_{k\in K} p^{\mathrm{str}}_i(k)e^{-ikx} \ , \tag{III.32}$$

where

$$p^{\mathrm{str}}_i(k) = \begin{cases} -i\frac{1}{\sqrt{2}}\left(a_{+,j}(k)^* - a_{-,j}(-k)\right) \ , & \text{for } k=0 \\ -i\sqrt{\frac{|k|}{2}}\left(a_{+,j}(k)^* - a_{-,j}(-k)\right) \ , & \text{for } k\in K, \text{ and } k\neq 0. \end{cases} \tag{III.33}$$

These time-zero fields satisfy the canonical relations $[\pi_i^{\rm str}(x), \varphi_{i'}^{\rm str}(y)] = -i\delta_{ii'}\delta(x-y)I$. The free Hamiltonian has the form

$$H_0^{b,{\rm str}} = \sum_{i=1}^{n} \left(\left(p_i^{\rm str}(0)^* p_i^{\rm str}(0) - \frac{1}{2} I \right) + \sum_{\substack{k \in K \\ k \neq 0}} |k| \left(a_{+,i}(k)^* a_{+,i}(k) + a_{-,i}(k)^* a_{-,i}(k) \right) \right) , \tag{III.34}$$

which can be expressed as the integral of the energy density

$$H_0^{b,{\rm str}} = \sum_{i=1}^{n} \int_{\mathcal{M}} : \overline{\pi}_i^{\rm str} \pi_i^{\rm str} + \sum_{j=1}^{s} \partial_j \overline{\varphi}_i^{\rm str} \partial_j \varphi_i^{\rm str} : dx \; . \tag{III.35}$$

The real time field is

$$\varphi_{{\rm RT},i}^{\rm str}(x,t) = \frac{1}{\sqrt{|\mathcal{M}|}} \left(q_i^{\rm str}(0) + p_i^{\rm str}(0)^* t + \sum_{\substack{k \in K \\ k \neq 0}} \frac{1}{(2|k|)^{1/2}} \left(a_{+,j}(k)^* e^{i|k|t} + a_{-,j}(-k) e^{-i|k|t} \right) e^{-ikx} \right) . \tag{III.36}$$

This is a solution to the wave equastion

$$\frac{\partial^2}{\partial t^2} \varphi_{\rm RT}^{\rm str}(x,t) = \nabla^2 \varphi_{\rm RT}^{\rm str}(x,t) \; , \tag{III.37}$$

satisfying the equal-time canonical commutation relations, and with initial data

$$\varphi_{\rm RT}^{\rm str}(x,0) = \varphi^{\rm str}(x) \; , \qquad \text{and} \qquad \left(\frac{\partial \varphi_{\rm RT}^{\rm str}}{\partial t} \right)(x,0) = \pi^{\rm str}(x)^* \; . \tag{III.38}$$

The momentum operator $P^{b,{\rm str}}$ and twist generator $J^{b,{\rm str}}$ have the same form as (III.28)–(III.29).

IV Interactions on the Circle

In this section we complete definition of the perturbed Hamiltonian $H^{b,{\rm cutoff}}$ introduced in (III.1). Here $\varphi^{\rm cutoff}$ denotes one of the three infra-red regularized fields of §III and $H_0^{b,{\rm cutoff}}$ denotes the corresponding free field Hamiltonian acting on $\mathcal{H}^{b,{\rm cutoff}}$. We consider the case $s = 1$ in this section, namely a spatial manifold $\mathcal{M} = S^1$ of length ℓ. In this section we denote the twist angle by $\chi = \{\chi_i\}$, dropping the second index.

IV.1 The Mass Perturbation

We begin with a quadratic interaction

$$H_M^{b,{\rm cutoff}} = H_0^{b,{\rm cutoff}} + M^2 \sum_{i=1}^{n} \int_0^{\ell} : \overline{\varphi}_i^{\rm cutoff} \varphi_i^{\rm cutoff} : dx \; . \tag{IV.1}$$

This Hamiltonian arises from the choice $W(z) = \frac{1}{2}M\sum_{i=1}^{n} z_i^2$. We are interested in the zero-point energy

$$\mathcal{E}_0^{\mathrm{cutoff},M} = \inf \mathrm{spectrum}(H_M^{b,\mathrm{cutoff}}) \,. \tag{IV.2}$$

We shall diagonalize the quadratic Hamiltonian (IV.1), and we show:

Proposition IV.1.1. *The three cutoff methods lead to the zero-point energy* $\mathcal{E}_0^{\mathrm{cutoff},M}$ *for the Hamiltonian* $H_M^{b,\mathrm{cutoff}}$ *of (IV.1) equal to*

$$\mathcal{E}_0^{\chi,M} = -\frac{1}{2}\sum_{i=1}^{n}\sum_{k\in K_i^\chi} \mu_M(k)\left(\left(\frac{\mu_M(k)}{|k|}\right)^{1/2} - \left(\frac{|k|}{\mu_M(k)}\right)^{1/2}\right)^2 , \tag{IV.3}$$

$$\mathcal{E}_0^{m,M} = -\frac{n}{2}\sum_{k\in K} \mu_{(M^2+m^2)^{1/2}}(k)\left(\left(\frac{\mu_{(M^2+m^2)^{1/2}}(k)}{\mu_m(k)}\right)^{1/2} - \left(\frac{\mu_m(k)}{\mu_{(M^2+m^2)^{1/2}}(k)}\right)^{1/2}\right)^2 , \tag{IV.4}$$

and

$$\mathcal{E}_0^{\mathrm{str},M} = -\frac{n}{2}(M-1)^2 - \frac{n}{2}\sum_{\substack{k\in K\\ k\neq 0}} \mu_M(k)\left(\left(\frac{\mu_M(k)}{|k|}\right)^{1/2} - \left(\frac{|k|}{\mu_M(k)}\right)^{1/2}\right)^2 . \tag{IV.5}$$

Here $\mu_M(k)$ *is defined in (III.20).*

Remark. For fixed $M > 0$, the function $\mu_M(k)\left(\left(\frac{\mu_M(k)}{|k|}\right)^{1/2} - \left(\frac{|k|}{\mu_M(k)}\right)^{1/2}\right)^2$ has the asymptotic behavior

$$\mu_M(k)\left(\left(\frac{\mu_M(k)}{|k|}\right)^{1/2} - \left(\frac{|k|}{\mu_M(k)}\right)^{1/2}\right)^2 \sim \begin{cases} |k|^{-3}, & |k|\to\infty \\ |k|^{-1}, & |k|\to 0 \end{cases} . \tag{IV.6}$$

As a consequence, each of the above zero-point energies in Proposition IV.1.1 is summable over k, and the corresponding Hamiltonian is bounded from below. This result extends to three space-time dimensions, but the zero-point energy diverges logarithmically in four space-time dimensions. Furthermore, for fixed M, the zero-point energy $\mathcal{E}_0^{\chi,M}$ diverges as $\chi\to 0$ with $M > 0$ fixed. Also, $\mathcal{E}_0^{m,M}$ diverges as $m \to 0$ with $M > 0$ fixed. On the other hand, $\mathcal{E}_0^{\mathrm{str},M}$ is well defined for fixed $M > 0$.

Proof. The momentum-k modes from the i^{th} component of the field that enter the mass-perturbation Hamiltonian only couple to other modes from the same component and with momentum $\pm k$. Thus we consider these modes separately. Their contribution to the twist-cutoff Hamiltonian is

$$\begin{aligned} H_i^{b,\chi}(k) &= |k|\left(a_{+,i}^\chi(k)^* a_{+,i}^\chi(k) + a_{-,i}^\chi(-k)^* a_{-,i}^\chi(-k)\right) \\ &\quad + \frac{M^2}{2|k|}\left(a_{+,i}^\chi(k)^* a_{+,i}^\chi(k) + a_{-,i}^\chi(-k)^* a_{-,i}^\chi(-k) + a_{+,i}^\chi(k) a_{-,i}^\chi(-k) + a_{+,i}^\chi(k)^* a_{-,i}^\chi(-k)^*\right) , \end{aligned} \tag{IV.7}$$

and

$$H^{b,\chi} = \sum_{i=1}^{n} \sum_{k\in K_i^\chi} H_i^{b,\chi}(k) \ . \tag{IV.8}$$

We rewrite the Hamiltonian (IV.7) in the form

$$H_i^{b,\chi}(k) = \mu_M(k) \left(A_{+,i}^\chi(k)^* A_{+,i}^\chi(k) + A_{-,i}^\chi(-k)^* A_{-,i}^\chi(-k) \right) + \mathcal{E}_0^{b,\chi}(M,k,i) \ , \tag{IV.9}$$

where $\mathcal{E}_0^{b,\chi}(M,k,i)$ is the zero-point energy for the modes under consideration. Thus

$$\mathcal{E}_0^{\chi,M} = \sum_{i=1}^{n} \sum_{k\in K_i^\chi} \mathcal{E}_0^{b,\chi}(M,k,i) \ , \tag{IV.10}$$

and

$$H^{b,\chi} = \sum_{i=1}^{n} \sum_{k\in K_i^\chi} \left(\mu_M(k) \left(A_{+,i}^\chi(k)^* A_{+,i}^\chi(k) + A_{-,i}^\chi(-k)^* A_{-,i}^\chi(-k) \right) + \mathcal{E}_0^{b,\chi}(M,k,i) \right) \ , \tag{IV.11}$$

We do this by making a canonical transformation depending on a parameter $\alpha = \alpha(M,|k|)$. For $k \in K_i^\chi$, define

$$A_{+,i}^\chi(k) = a_{+,i}^\chi(\pm k)\cosh\alpha + a_{-,i}^\chi(-k)^*\sinh\alpha \ , \ \text{and} \ A_{-,i}^\chi(-k) = a_{-,i}^\chi(-k)\cosh\alpha + a_{+,i}^\chi(k)^*\sinh\alpha \ . \tag{IV.12}$$

The new canonical variables satisfy

$$[A_{+,i}^\chi(k), A_{-,j}^\chi(k')^\#] = 0 \ , \ \text{and} \ [A_{\pm,i}^\chi(k), A_{\pm,j}^\chi(k')^*] = \delta_{ij}\delta_{kk'} I \ . \tag{IV.13}$$

Comparing (IV.7) with (IV.9) we find that the parameter α must satisfy

$$\cosh\alpha + \sinh\alpha = u \ , \qquad \text{and} \qquad \cosh\alpha - \sinh\alpha = u^{-1} \ , \tag{IV.14}$$

where $u = \sqrt{\frac{\mu_M(k)}{|k|}}$. This yields

$$\alpha = \operatorname{arc}\cosh\left(\frac{1}{2}(u+u^{-1})\right) \ . \tag{IV.15}$$

Also this comparison leads to

$$\mathcal{E}_0(M,k,i) = -\frac{1}{2}\mu_M(k)\left(u - \frac{1}{u}\right)^2 \ , \tag{IV.16}$$

from which (IV.3) follows. The mass cutoff can be handled in a similar fashion, leading to (IV.4).

Finally we treat the string method cutoff, and this also can be handled in a similar fashion. The one difference from the above concerns the constant Fourier modes, which undergoes a different canonical transformation. In fact, the $k = 0$ mode contribution to the Hamiltonian for the i^{th} component is not the $k \to 0$ limit of (IV.7), but rather it is

$$H^{b,\text{str}}(0) = \sum_{i=1}^{n} \left(p_i^* p_i + M^2 q_i^* q_i - \frac{1}{2}(1+M^2) \right) = \sum_{i=1}^{n} : p_i^* p_i + M^2 q_i^* q_i : . \tag{IV.17}$$

The quadratic term in p_i arises from the free Hamiltonian (III.35), the quadratic term in q_i arises from the quadratic interaction (IV.1), and the constant has a contribution from each.

$$q_i = \frac{1}{\sqrt{2}} \left(a_{+,i}^*(0) + a_{-,i}(0) \right) , \quad \text{and} \quad p_i = \frac{-i}{\sqrt{2}} \left(a_{+,i}(0) - a_{-,i}^*(0) \right) . \tag{IV.18}$$

Define the canonical annihilation variables

$$A_{\pm,i} = a_{\pm,i}(0) \cosh\alpha + a_{\mp,i}^*(0) \sinh\alpha , \qquad \text{with} \qquad \alpha = \alpha(M) = \frac{1}{2} \ln M , \tag{IV.19}$$

in terms of which the identity

$$H^{b,\text{str}}(0) = M \sum_{i=1}^{n} \left(A_{+,i}^* A_{+,i} + A_{-,i}^* A_{-,i} \right) - \frac{n}{2}(M-1)^2 , \tag{IV.20}$$

completes the proof of Proposition VI.1.1.

Further elaboration of the diagonalization leads to:

Proposition IV.1.2. *The operator $H_M^{b,\text{cutoff}}$ has the following properties:*

a. *For $M > 0$, the operators $H_M^{b,\text{cutoff}}$ are bounded from below and essentially self adjoint on $\mathcal{D}$.*

b. *The heat kernel $\exp(-\beta H_M^{b,\text{cutoff}})$ commutes with $U(\sigma, \theta)$.*

c. *The ground state Ω_{vac} for $H_M^{b,\text{cutoff}}$ is unique, and it satisfies*

$$U(\sigma,\theta)\Omega_{\text{vac}} = \Omega_{\text{vac}} . \tag{IV.21}$$

d. *For $\beta > 0$ the heat kernel is trace class, and*

$$\mathfrak{Z}^{\text{cutoff},M}(\mathcal{T}) = \text{Tr}_{\mathcal{H}^{b,\text{cutoff}}} \left(U(\sigma,\theta)^* e^{-\beta H_M^{b,\text{cutoff}}} \right) = \prod_{i=1}^{n} \prod_{k \in K_i^{\text{cutoff}}} \frac{e^{-\beta \mathcal{E}_0^{b,\text{cutoff}}(M,k,i)}}{|1 - \gamma_i^{\mathcal{T}}(k)|^2} , \tag{IV.22}$$

where

$$\gamma_i^{\mathcal{T}}(k) = \begin{cases} e^{-i\theta\Omega_i - i\sigma k - \beta\mu_M(k)} , & \text{for twist and string regularization,} \\ e^{-i\theta\Omega_i - i\sigma k - \beta\mu_{M'}(k)} , & M' = \sqrt{M^2 + m^2} \text{ for the mass regularization.} \end{cases} \tag{IV.23}$$

Proof. The proof of essential self-adjointness claimed in (a) is a consequence of the representation (IV.8) of the Hamiltonian as a sum of mutually commuting Hamiltonians. Each $H_i^{b,\chi}(k)$ is essentially self adjoint as a consequence of the standard arguments. See for example [8]. In addition, we have an explicit diagonalization of $H_M^{b,\text{cutoff}}$. We give the details for the twist field; the other cases of the massive field and the string field are similar. We use the same representation as in the proof of Proposition IV.1.1. This procedure also diagonalizes $P^{b,\text{cutoff}}$ and $J^{b,\text{cutoff}}$, and using this analysis we show that the ground state of $H_M^{b,co}$ is annihilated by $P^{b,\text{cutoff}}$ and by $J^{b,\text{cutoff}}$, proving (b) and (c).

Recall the definition (II.12) of the coordinates $q_i^\chi(k)$. With $A_{\pm,i}^\chi(k)$ defined in (IV.12), define the related coordinates

$$Q_i^\chi(k) = \frac{1}{\sqrt{2\mu_M(k)}} \left(A_{+,i}^\chi(k)^* + A_{-,i}^\chi(-k) \right) . \tag{IV.24}$$

Also denote the number of $A_{\pm,i}^\chi(k)$ quanta as

$$N_{\pm,i}^{A^\chi}(\pm k) = A_{+,i}^\chi(\pm k)^* A_{+,i}^\chi(\pm k) , \tag{IV.25}$$

where we use the superscript A^χ to denote the choice of canonical variables. We denote the corresponding number operators for the $a_{\pm,i}^\chi(\pm k)$ quanta as

$$N_{\pm,i}^{a^\chi}(\pm k) = a_{+,i}^\chi(\pm k)^* a_{+,i}^\chi(\pm k) . \tag{IV.26}$$

We note two important algebraic identities whose proof are straightforward consequences of (IV.12)–(IV.15):

Lemma IV.1.3. *The choice (IV.12) of canonical variables yields for $k \in K_i^\chi$,*

$$q_i^\chi(k) = Q_i^\chi(k) , \tag{IV.27}$$

and

$$N_{+,i}^{A^\chi}(k) - N_{-,i}^{A^\chi}(-k) = N_{+,i}^{a^\chi}(k) - N_{-,i}^{a^\chi}(-k) . \tag{IV.28}$$

As a result of (IV.28),

$$P^{b,\chi} = \sum_{i=1}^n \sum_{k\in K_i^\chi} k \left(A_{+,i}^\chi(k)^* A_{+,i}^\chi(k) - A_{-,i}^\chi(-k)^* A_{-,i}^\chi(-k) \right) , \tag{IV.29}$$

and

$$J^{b,\chi} = \sum_{i=1}^n \sum_{k\in K_i^\chi} \Omega_i \left(A_{+,i}^\chi(k)^* A_{+,i}^\chi(k) - A_{-,i}^\chi(-k)^* A_{-,i}^\chi(-k) \right) . \tag{IV.30}$$

Thus both $P^{b,\chi}$ and $J^{b,\chi}$ have similar expansions when expressed in terms of the A^χ variables as in the a^χ variables.

A standard argument in quantum theory, for example Corollary 3.3.4 of [1], ensures that the ground state $\Omega_{\rm vac}$ of $H_M^{b,\chi}$ is unique. Each $H_{M,i}^{b,\chi} - \mathcal{E}_0(M,k,i) \geq 0$, and the sum of these operators has the ground state $\Omega_{\rm vac}$ with eigenvalue zero. The wave function for the eigenstate of this mode has the form $c(k)e^{-|Q_i^\chi(k)|^2}$, where $c(k)$ is a normalization constant. Thus each $H_{M,i}^{b,\chi}$ satisfies $H_{M,i}^{b,\chi}\Omega_{\rm vac} = \mathcal{E}_0(M,k,i)\Omega_{\rm vac}$, or

$$A_{\pm,i}^\chi(\pm k)^* A_{\pm,i}^\chi(\pm k)\Omega_{\rm vac} = N_{\pm,i}^{A\chi}(\pm k)\Omega_{\rm vac} = 0 \ . \tag{IV.31}$$

Therefore the ground state vector $\Omega_{\rm vac}$ of $H^{b,\chi}$ satisfies

$$H^{b,\chi}\Omega_{\rm vac} = \mathcal{E}_0^{b,\chi}\Omega_{\rm vac} \ , \qquad P^{b,\chi}\Omega_{\rm vac} = 0 \ , \qquad \text{and} \qquad J^{b,\chi}\Omega_{\rm vac} = 0 \ . \tag{IV.32}$$

As a consequence, $\Omega_{\rm vac}$ is invariant under the symmetry group $U(\sigma,\theta)$,

$$U(\sigma,\theta)\Omega_{\rm vac} = \Omega_{\rm vac} \ , \tag{IV.33}$$

which is the normalization required in the general discussion of twist positivity [2].

This justifies our use of the new canonical coordinates (IV.19) to simultaneously diagonalize $H^{b,\chi}$, $P^{b,\chi}$, and $J^{b,\chi}$. The orthonormal eigenstates have the form

$$\left(\prod_{1\leq i\leq n} \prod_{k\in K_i^\chi} \frac{1}{\sqrt{n_+(i,k)!n_-(i,k)!}} A_{+,i}^\chi(k)^{*n_+(i,k)} A_{-,i}^\chi(-k)^{*n_-(i,k)} \right) \Omega_{\rm vac} \ , \tag{IV.34}$$

where only a finite number of the $n_\pm(i,k) \in \mathbb{Z}_+$ are nonzero. This also justifies using the proof of Proposition VI.1.1 of [2], modified to take into account the fact that the zero-point energy does not vanish. Hence the proof of Proposition IV.1.2 is complete.

Having established the trace class property of the heat kernel, we define the corresponding normalized functional

$$\langle \ \cdot \ \rangle_{\mathcal{T}}^{\text{cutoff},M} = \frac{\operatorname{Tr}_{\mathcal{H}^{b,\text{cutoff}}}\left(\ \cdot \ U(\sigma,\theta)^* e^{-\beta H_M^{b,\text{cutoff}}} \right)}{\operatorname{Tr}_{\mathcal{H}^{b,\text{cutoff}}}\left(U(\sigma,\theta)^* e^{-\beta H_M^{b,\text{cutoff}}} \right)} \ . \tag{IV.35}$$

In this subsection we let $\mathcal{T} = \{\sigma,\theta,\beta\}$, while 'cutoff' denotes χ in the case of twist fields, m in the case of massive fields, and 'string' in case of the string field. Our main observation in this section is

Proposition IV.1.4. *Let $M > 0$, let $\beta > 0$ and let $\mathcal{T}$ be fixed.*

a. *The functional (IV.35) is a Gaussian function of time-ordered fields. The expectation of each field vanishes, $\left\langle \varphi^{\text{cutoff}} \right\rangle_{\mathcal{T}}^{\text{cutoff},M} = 0$. The pair correlation matrix for the twist fields equals*

$$\left\langle \left(\overline{\varphi}_i^\chi(x,t)\varphi_j^\chi(y,s) \right)_+ \right\rangle_{\mathcal{T}}^{\chi,M} = \delta_{ij}\left(\Delta_{\chi,\mathcal{T}} + M^2 \right)^{-1}(x-y,t-s) \ . \tag{IV.36}$$

Here $\Delta_{\chi,\mathcal{T}}$ denotes the Laplacian (II.84) on $\mathcal{S}_{\mathcal{T}}(\mathbb{T}^2)$ with twist relations

$$f_i(x+\ell,t) = e^{-i\chi_i}f_i(x,t) \qquad \textit{and} \qquad f_i(x,t+\beta) = e^{-i\Omega_i\theta}f_i(x+\sigma,t) \ . \tag{IV.37}$$

b. *The pair correlation function for the massive cutoff equals*

$$\left\langle \left(\overline{\varphi}_i^m(x,t)\varphi_j^m(y,s)\right)_+ \right\rangle_{\mathcal{T}}^{m,M} = \delta_{ij} \left(\Delta_{\mathcal{T}} + m^2 + M^2\right)^{-1} (x-y, t-s) \ . \tag{IV.38}$$

Here $\Delta_{\mathcal{T}}$ *denotes the Laplacian (II.84) on* $\mathcal{S}_{\mathcal{T}}(\mathbb{T}^2)$ *with twist relations*

$$f_i(x+\ell, t) = f_i(x,t) \quad \text{and} \quad f_i(x, t+\beta) = e^{-i\Omega_i\theta} f_i(x+\sigma, t) \ . \tag{IV.39}$$

c. *The pair correlation function in the string case equals*

$$\left\langle \left(\overline{\varphi}_i(x,t)\varphi_j(y,s)\right)_+ \right\rangle_{\mathcal{T}}^{\mathrm{str},M} = \delta_{ij} \left(\Delta_{\mathcal{T}} + M^2\right)^{-1} (x-y, t-s) \ . \tag{IV.40}$$

Here $\Delta_{\mathcal{T}}$ *denotes the same Laplacian as in (b).*

d. *For* $M > 0$ *and fixed, the limit of the twist-field pair correlation matrix exists as* $\chi \to 0$*. The limit of the massive-field pair correlation matrix exists as* $m \to 0$*. These limits both exist in the sense of distributions on* $(\otimes_{i=1}^n C^\infty(\Sigma))'$*, and entail the convergence of the corresponding field theories in the sense defined in [9]. Both limits agree with the string-field pair correlation function (IV.40).*

Proof. We establish the fact that the functionals (IV.35) are Gaussian using the same method as we establish Proposition II.3.4, namely the proof of Propositions II.3 and VI.2 of [2]; we omit the details. This method also allows us to evaluate the expectations (IV.40). It is clear that for each example the trace factors over Hilbert spaces associated with each component of the field, and over momenta as well, if we retain the modes involving $\pm k$ in the same factor. In terms of the pair correlation matrix, this means that the matrix is diagonal.

Using Lemma IV.1.3, we express the field $\varphi_i^\chi(x)$, originally defined in (II.11), in terms of the new canonical variables $A_{\pm,i}^\chi(k)$. In the case $s = 1$ we have

$$\varphi_i^\chi(x) = \frac{1}{\ell^{1/2}} \sum_{k \in K_i^\chi} q_i^\chi(k) e^{-ikx} = \frac{1}{\ell^{1/2}} \sum_{k \in K_i^\chi} Q_i^\chi(k) e^{-ikx} \ . \tag{IV.41}$$

We can compute the pair correlation matrix in the basis of new creation and annihilation operators, and in this basis we can simultaneously diagonalize the Hamiltonian (IV.9), the momentum operator $P^{b,\chi}$, and the twist generator $J^{b,\chi}$. For example, for $0 \le t < s \le \beta$, and with imaginary time propagation,

$$\left\langle \left(q_i(k)^*(t) q_i(k)(s)\right)_+ \right\rangle_{\mathcal{T}}^{\chi,M} e^{ik(x-y)} = \langle q_i(k)^*(t) q_i(k)(s) \rangle_{\mathcal{T}}^{\chi,M} e^{ik(x-y)} = \langle Q_i(k)^*(t) Q_i(k)(s) \rangle_{\mathcal{T}}^{\chi,M} e^{ik(x-y)} \tag{IV.42}$$

on the subspace spanned by the degrees of freedom $a_{\pm,i}(\pm k)$ or $A_{\pm,i}(\pm k)$. The full pair correlation matrix (II.67), multiplied by ℓ, is the sum of (IV.42) over $k \in K_i^\chi$. Following the proof of Theorem VI.8 c–d of [2], we obtain

$$\left\langle \left(\overline{\varphi}_i^\chi(x,t) \varphi_j^\chi(y,s) \right)_+ \right\rangle_{\mathcal{T}}^{\chi,M} = \delta_{ij} \left(\Delta_{\{\chi,\mathcal{T}\}} + M^2 \right)^{-1} (x-y, t-s) \ . \tag{IV.43}$$

Here we use $\Delta_{\{\chi,\mathcal{T}\}}$ to denote the Laplace operator (II.84) of §II.4. As a bounded operator on $L^2(\Sigma) = L^2(S^1 \times S^1)$, the resolvent $\left(\Delta_{\chi,\mathcal{T}} + M^2\right)^{-1}$ converges weakly as $\chi \to 0$. Thus it converges in the sense of distributions to (IV.40).

For the massive field, we clearly can carry out exactly the same argument, to obtain (IV.40) as $m \to 0$ with $M > 0$ fixed. We do not give the details. In the string case, the zero-momentum modes also satisfy

$$q_i^{\rm str}(0) = \frac{1}{(2M)^{1/2}} \left(A_{+,i}^* + A_{-,i} \right) \ , \tag{IV.44}$$

and

$$A_{+,i}^* A_{+,i} - A_{-,i}^* A_{-,i} = a_{+,i}(0)^* a_{+,i}(0) - a_{-,i}(0)^* a_{-,i}(0) \ . \tag{IV.45}$$

This gives the desired representations for the constant modes, and completes the proof of Proposition IV.1.3.

IV.2 Stable Nonlinearities

In this section we extend the construction to nonlinear interactions arising from the potentials $|\text{grad}\, W|^2$ introduced in (III.8). We take the nonlinear term in H^b to be

$$V(W) = \int_0^\ell : |\text{grad}\, W(\varphi^{\rm cutoff}(x))|^2 : \, dx = \sum_{j=1}^n \int_0^\ell : |W_j(\varphi^{\rm cutoff}(x))|^2 : \, dx \ , \tag{IV.46}$$

where W is a polynomial introduced in §III.2. We require estimates on $V(W)$ that are uniform at high-frequency. To obtain these, we introduce a family of ultra-violet mollifiers $\mathcal{K}_{\Lambda\,,\,j}(x)$, parameterized by a positive number Λ. The mollifiers act on the time-zero fields $\varphi^{\rm cutoff}$ by convolution, and they converge to the identity as $\Lambda \to \infty$. Denote the doubly-regularized bosonic field by

$$\varphi_{\Lambda,j}(x) = \int_0^\ell \mathcal{K}_{\Lambda\,,\,j}(x-y)\, \varphi_j^{\rm cutoff}(y)\, dy \ . \tag{IV.47}$$

We construct the mollifier $\mathcal{K}_{\Lambda\,,\,j}(x)$ in the following manner. Let $\mathcal{S}_j = \mathcal{S}^{\Omega_j \phi}$ denote the linear space of C^∞ functions on the circle that satisfy the twist relation $f(x+\ell) = e^{i\Omega_j\phi} f(x)$. Let $\mathcal{S}_{-j} = \mathcal{S}^{-\Omega_j\phi}$ denote the complex conjugate space. The space $\mathcal{S}_j$ is a subspace of the space of generalized functions $(\mathcal{S}_{-j})'$ dual to $\mathcal{S}_{-j}$. Take a real, even, C^∞-function $\hat{\mathcal{K}}(k)$ defined for $k \in \mathbb{R}$, with the additional property that it satisfies the bounds

$$\frac{1}{(1+k^2)^\epsilon} \le \hat{\mathcal{K}}(k) \le \hat{\mathcal{K}}(0) = 1 \,, \tag{IV.48}$$

where $0 < \epsilon$ is a given constant. We first used a lower bound of this sort in [10], where we called it "slow decrease at infinity" or *sdi*. It is convenient to choose

$$\hat{\mathcal{K}}(k) = \frac{1}{(1+k^2)^\epsilon} \,. \tag{IV.49}$$

Define the family of kernels by the Fourier representations

$$\mathcal{K}_{\Lambda\,,j}(x) = \frac{1}{\ell} \sum_{k\in K_j^\chi} \hat{\mathcal{K}}(k/\Lambda) e^{-ikx} \,, \qquad \text{where} \qquad K_j^\chi = \{\ell k_j \in 2\pi\mathbb{Z} - \Omega_j \phi\} \,, \tag{IV.50}$$

and where the sum converges in the sense of $(\mathcal{S}_{-j})'$. Consequently, the kernels $\mathcal{K}_{\Lambda\,,j}$ act as convolution operators on the fields $\varphi_{\Lambda,j} \in (\mathcal{S}_{-j})'$, mapping $(\mathcal{S}_{-j})'$ continuously into $(\mathcal{S}_{-j})'$, and

$$\varphi_{\Lambda,j}(x) = \int_0^\ell \mathcal{K}_{\Lambda\,,j}(x-y)\varphi^{\text{cutoff}}_{\Lambda,j}(y)dy = \frac{1}{\sqrt{|\ell|}} \sum_{k\in K_i^\chi} \frac{1}{(2|k|)^{1/2}} \left(a^\chi_{+,i}(k)^* + a^\chi_{-,i}(-k)\right) \hat{\mathcal{K}}(k/\Lambda) e^{-ikx} \,. \tag{IV.51}$$

The family $\{\mathcal{K}_{\Lambda\,,j}\}$ converges as a sequence of convolution operators on $(\mathcal{S}_{-j})'$ to the Dirac measure δ concentrated at the origin,

$$\lim_{\Lambda\to\infty} \mathcal{K}_{\Lambda\,,j} = \delta \,. \tag{IV.52}$$

This choice of mollifier allows us to generalize constructive field theory methods (originally established for local perturbations of H_0^b) to certain bi-local perturbations of H_0^b, namely

$$V_\Lambda^{\text{nonlocal}}(W) = \sum_{j=1}^n \int_0^\ell \int_0^\ell :\overline{W_j(\varphi^{\text{cutoff}}_{\Lambda,j}(x))}\; v_{\Lambda,j}(x-y)\; W_j(\varphi^{\text{cutoff}}_{\Lambda,j}(y)):\; dx\,dy \,. \tag{IV.53}$$

Here the bi-local kernel $v_{\Lambda,j}(x-y)$ is an approximate Dirac measure

$$v_{\Lambda,j}(x) = e^{i(1-2\Omega_j)\phi x/\ell} \left(\frac{1}{\ell} \sum_{k\in K_j^\chi} |\hat{\mathcal{K}}(k/\Lambda)|^2 \, e^{-ikx} \right) \,, \tag{IV.54}$$

that is a distribution of positive type. We introduce this particular kernel because the bi-local potential (IV.53), without normal ordering, and with $v_{\Lambda,j}(x)$ of the form (IV.54), arises from introducing high-frequency mollifiers into a supersymmetric interaction. We show in Proposition VII.1.3 that the kernel $v_{\Lambda,j}(x)$ arises as the bosonic part of the Hamiltonian (VII.53).

Let us define

$$H_\Lambda^{b,\text{cutoff}}(W) = H_0^{b,\text{cutoff}} + V_\Lambda^{\text{nonlocal}}(W) \,, \tag{IV.55}$$

with $V_\Lambda^{\text{nonlocal}}(W)$ defined in (IV.53), and with domain $\mathcal{D}_\infty$. The methods of [10] immediately lead to the following.

Proposition IV.2.1 *Let W be a holomorphic, quasi-homogeneous polynomial that satisfies the bounds (III.17) and (III.18). Let $\mathcal{K}_{\Lambda,j}$ denote the sdi mollifier (IV.50), with $0 < \epsilon = \epsilon(W)$ sufficiently small, and with $0 < \Lambda < \infty$. Let $V_\Lambda^{\mathrm{nonlocal}}(W)$ be given by (IV.53), and let the Hamiltonian $H_\Lambda^{b,\mathrm{cutoff}}(W)$ be defined by (IV.55). Then,*

a. *The operator $H_\Lambda^{b,\mathrm{cutoff}}(W)$ is essentially self adjoint.*

b. *For $\beta > 0$, the heat kernel $e^{-\beta H_\Lambda^{b,\mathrm{cutoff}}(W)}$ is trace class.*

V Dirac Twist Fields on a Circle

V.1 Spinors

There are two sorts of fermi field on the circle; they are neutral (or Majorana) fields and charged (or Dirac) fields. The twist condition applies naturally to charged fields, so as in the bosonic case, we introduce Dirac fields directly.

Let $\mathrm{Mat}_2(\mathbb{C})$ denote the space of 2×2-complex matrices with the standard transpose and hermitian adjoint denoted by S^{T} and S^* respectively,

$$(S^{\mathrm{T}})_{ij} = S_{ji} , \qquad \text{and} \qquad (S^*)_{ij} = \overline{S_{ji}} . \tag{V.1}$$

Also let $\overline{S}$ denote the complex conjugate matrix,

$$(\overline{S})_{ij} = (S^{*\,\mathrm{T}})_{ij} = \overline{S_{ij}} . \tag{V.2}$$

We use an explicit representation for the Dirac matrices $\gamma^0, \gamma^1 \in \mathrm{Mat}_2(\mathbb{C})$, where γ^0 is hermitian and γ^1 is skew-hermitian,

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} . \tag{V.3}$$

Define

$$\sigma = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{V.4}$$

With $\{A, B\} = AB + BA$, these matrices satisfy

$$\{\gamma^i, \gamma^j\} = 2g^{ij} I , \qquad \text{and} \qquad \sigma\gamma^i = -\gamma^i\sigma , \tag{V.5}$$

where $g^{00} = 1$, $g^{11} = -1$, and $g^{01} = g^{10} = 0$. These conventions are consistent with much of the particle physics literature, and differ from our previous papers.

Let these matrices act on *spinors* $\eta \in \mathbb{C}^2$ that we denote

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} , \qquad \text{with the hermitian adjoint} \qquad \eta^* = (\, \eta_1^*, \eta_2^* \,) \ . \tag{V.6}$$

On components, η_j^* denotes complex conjugation. Following physics notation, we also define an adjoint spinor $\overline{\eta}$ by

$$\overline{\eta} = \eta^* \gamma^0 \ , \qquad \text{or in components} \qquad \overline{\eta} = (\, i\eta_2^*, \ -i\eta_1^* \,) \ . \tag{V.7}$$

(In previous sections we use $\overline{a}$ to denote the complex conjugate of $a \in \mathbb{C}$. However, we believe that no confusion will occur in following the physics convention to denote adjoint spinors by $\overline{\eta}$.)

Another standard involution $\eta \to \eta^{\mathrm{c}}$ on spinors is charge conjugation,

$$\eta^{\mathrm{c}} = \eta^{*\,\mathrm{T}} = \begin{pmatrix} \eta_1^* \\ \eta_2^* \end{pmatrix} \ . \tag{V.8}$$

This can also be written

$$\eta^{\mathrm{c}} = C\,\overline{\eta}^{\mathrm{T}} = C\left(\gamma^0\right)^{\mathrm{T}} \eta^{*\,\mathrm{T}} \ , \tag{V.9}$$

where C is called the charge conjugation matrix. In (V.8) we make the choice

$$C\left(\gamma^0\right)^{\mathrm{T}} = I \ , \tag{V.10}$$

as discussed further in §V.2.

We call these elements of $\mathbb{C}^2$ spinors, because of their transformation under Lorentz boosts. The Lorentz 'boost' is generated by $\sigma = \frac{1}{2}[\gamma^0, \gamma^1] = \gamma^0\gamma^1$, and we define the boost as the $SL(2, \mathbb{C})$-transformation

$$\eta \to \eta' = e^{\sigma\phi/2}\eta = \begin{pmatrix} e^{\phi/2} & 0 \\ 0 & e^{-\phi/2} \end{pmatrix} \eta \ , \tag{V.11}$$

where $\phi \in \mathbb{R}$ is a parameter (hyperbolic angle). The adjoint spinor $\overline{\eta}$ combines with a spinor ζ to form a Lorentz scalar,

$$\overline{\eta}\,\zeta = \eta^*\gamma^0\zeta = i(\eta_2^*\,\zeta_1 - \eta_1^*\,\zeta_2) \ . \tag{V.12}$$

The combination (V.12) is a scalar in the sense that

$$\overline{\eta'}\,\zeta' = \overline{\eta}\,\zeta \ . \tag{V.13}$$

The spinor $\overline{\eta}$ also combines with ζ and the Dirac matrices to form the components of a 2-vector

$$\overline{\eta}\,\gamma^0\,\zeta = \eta^*\,\zeta = \eta_1^*\,\zeta_1 + \eta_2^*\,\zeta_2 \ , \qquad \text{and} \qquad \overline{\eta}\,\gamma^1\,\zeta = \eta^*\sigma\,\zeta = \eta_1^*\,\zeta_1 - \eta_2^*\,\zeta_2 \ . \tag{V.14}$$

These quantities transform under Lorentz boosts according to the hyperbolic rotation

$$\begin{pmatrix} \overline{\eta'}\,\gamma^0\,\zeta' \\ \overline{\eta'}\,\gamma^1\,\zeta' \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \cosh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \overline{\eta}\,\gamma^0\,\zeta \\ \overline{\eta}\,\gamma^1\,\zeta \end{pmatrix} \ . \tag{V.15}$$

We also consider n-copies of such spinors $\eta_{\alpha,i}$ with components $\eta_{\alpha,i}$. Here the second subscript i labels the copy, while the first subscript α labels the component within the i^{th} copy. In that case, we also use the notation,

$$\overline{\eta}\,\zeta = i\sum_{i=1}^{n}(\eta_{2,i}^{*}\,\zeta_{1,i} - \eta_{1,i}^{*}\,\zeta_{2,i})\;, \qquad \text{and} \qquad \overline{\eta}\,\gamma^{j}\,\zeta = \sum_{i=1}^{n}\overline{\eta}_{i}\,\gamma^{j}\,\zeta_{i} = \sum_{i=1}^{n}\sum_{\alpha,\beta=1}^{2}\eta_{\alpha,i}^{*}\,\left(\gamma^{0}\gamma^{j}\right)_{\alpha\beta}\,\zeta_{\beta,i}\;. \tag{V.16}$$

In order to define free fermion quantum fields, we introduce the fermionic Hilbert space. As in the case of bosons, the one particle space depends on the twists. We define the time-zero, Dirac quantum field $\psi^{\chi}(x)$ by their Fourier representations. The components will be $\{\psi^{\chi}_{\alpha,i}(x)\}$, where the index $1 \le i \le n$ designates a copy (as for the bosonic felds) and the index $1 \le \alpha \le 2$ labels the components of the particular copy. The twist angles for fermions may be chosen independently from the twist angles for bosons. Thus we let χ denote the set of bosonic and fermionic twists,

$$\chi = \left(\chi^{b}, \chi^{f}\right)\;, \qquad \text{where} \qquad \chi^{f} = \left(\chi^{f}_{\alpha,i} : 1 \le \alpha \le 2, 1 \le i \le n\right)\;. \tag{V.17}$$

The fermionic twists depend on both i and α. We choose fermionic twist angles such that

$$e^{i\chi^{f}_{\alpha,i}} \neq 1\;, \tag{V.18}$$

for all $1 \le \alpha \le 2$ and $1 \le i \le n$. Then the distinct components of the field also involve *non-zero* momenta. We define the fields and their time dependence in such a fashion that they satisfy the holonomy relations

$$\psi^{\chi}_{\alpha,i}(x+\ell,t) = e^{i\chi^{f}_{\alpha,i}}\,\psi^{\chi}_{\alpha,i}(x,t)\;. \tag{V.19}$$

We begin by introducing momentum sets for the components of the fields,

$$K^{\chi^{f}}_{\alpha,i} = \{k : \ell k \in 2\pi\mathbb{Z} - \chi^{f}_{\alpha,i}\}\;. \tag{V.20}$$

The condition (V.18) ensures that

$$0 \notin K^{\chi^{f}}_{\alpha,i}\;. \tag{V.21}$$

It is also natural to introduce momentum sets for the $\pm$-modes of the creation and annihilation operators. We use $b^{\chi^{f}}_{+,i}(k)$ for $k \in K^{\chi^{f}}_{+,i}$, and $b^{\chi^{f}}_{-,i}(-k)$ for $k \in K^{\chi^{f}}_{-,i}$. Here

$$K^{\chi^{f}}_{+,i} = \left\{k : \left\{k > 0 \;\text{ and }\; k \in K^{\chi^{f}}_{1,i}\right\} \cup \left\{k < 0 \;\text{ and }\; k \in K^{\chi^{f}}_{2,i}\right\}\right\}\;, \tag{V.22}$$

and

$$K^{\chi^{f}}_{-,i} = \left\{k : \left\{k < 0 \;\text{ and }\; k \in K^{\chi^{f}}_{1,i}\right\} \cup \left\{k > 0 \;\text{ and }\; k \in K^{\chi^{f}}_{2,i}\right\}\right\}\;. \tag{V.23}$$

We can invert these relations; for example,

$$K^{\chi^{f}}_{1,i} = \left\{k : \left\{k > 0 \;\text{ and }\; k \in K^{\chi^{f}}_{+,i}\right\} \cup \left\{k < 0 \;\text{ and }\; k \in K^{\chi^{f}}_{-,i}\right\}\right\}\;. \tag{V.24}$$

The one-particle Hilbert space is

$$\mathcal{K}^{\chi^f} = \bigoplus_{i=1}^{n} \Big(l_2(K^{\chi^f}_{+,i}) \oplus l_2(-K^{\chi^f}_{-,i}) \Big) \ . \tag{V.25}$$

The Fock space $\mathcal{H}^{f,\chi}$ is the skew tensor algebra over $\mathcal{K}^{\chi^f}$,

$$\mathcal{H}^{f,\chi} = \exp_{\wedge} \mathcal{K}^{\chi^f} \ , \tag{V.26}$$

where $\wedge$ denotes the skew-symmetric tensor product. On this Hilbert space, we define two independent sets of canonical creation opertors on this Fock space.

The creation and annihilation operators satisfy the canonical anti-commutation relations (the CAR)

$$\{b^{\chi^f}_{+,i}(k), b^{\chi^f}_{+,i'}(k')\} = 0 \ , \qquad \text{and} \qquad \{b^{\chi^f}_{+,i}(k), b^{\chi^f}_{-,i'}(-k')^{\#}\} = 0 \ , \tag{V.27}$$

as well as

$$\{b^{\chi^f}_{+,i}(k), b^{\chi^f}_{+,i'}(k')^{*}\} = \delta_{ii'}\delta_{k,k'} I \ , \tag{V.28}$$

where $b^{\chi^f}_{+,i}(k)^{\#}$ denotes either $b^{\chi^f}_{+,i}(k)$ or $b^{\chi^f}_{+,i}(k)^{*}$, and

$$\{b^{\chi^f}_{-,i}(-k), b^{\chi^f}_{-,i'}(-k')^{*}\} = \delta_{ii'}\delta_{k,k'} I \ , \tag{V.29}$$

We express the fields in terms of their Fourier representation,

$$\psi^{\chi}_{1,i}(x) = \frac{1}{\sqrt{\ell}} \sum_{\substack{k>0 \\ k\in K^{\chi^f}_{+,i}}} b^{\chi^f}_{+,i}(k)^{*} e^{-ikx} + \frac{1}{\sqrt{\ell}} \sum_{\substack{k<0 \\ k\in K^{\chi^f}_{-,i}}} b^{\chi^f}_{-,i}(-k) e^{-ikx} = \frac{1}{\sqrt{\ell}} \sum_{k\in K^{\chi^f}_{1,i}} \xi^{\chi}_{1,i}(k) e^{-ikx} \ , \tag{V.30}$$

and

$$\psi^{\chi}_{2,i}(x) = \frac{-i}{\sqrt{\ell}} \sum_{\substack{k<0 \\ k\in K^{\chi^f}_{+,i}}} b^{\chi^f}_{+,i}(k)^{*} e^{-ikx} + \frac{i}{\sqrt{\ell}} \sum_{\substack{k>0 \\ k\in K^{\chi^f}_{-,i}}} b^{\chi^f}_{-,i}(-k) e^{-ikx} = \frac{1}{\sqrt{\ell}} \sum_{k\in K^{\chi^f}_{2,i}} \xi^{\chi}_{2,i}(k) e^{-ikx} \ . \tag{V.31}$$

Here $\xi^{\chi}_{\alpha,i}(k)$ denote fermionic coordinates. Explicitly,

$$\xi^{\chi}_{1,i}(k) = \begin{cases} b^{\chi^f}_{+,i}(k)^{*} \ , & \text{with } k \in K^{\chi^f}_{+,i} \text{ if } k>0 \\ b^{\chi^f}_{-,i}(-k) \ , & \text{with } k \in K^{\chi^f}_{-,i} \text{ if } k<0 \end{cases} \ , \tag{V.32}$$

and

$$\xi^{\chi}_{2,i}(k) = \begin{cases} i b^{\chi^f}_{-,i}(-k) \ , & \text{with } k \in K^{\chi^f}_{-,i} \text{ if } k>0 \\ -i b^{\chi^f}_{+,i}(k)^{*} \ , & \text{with } k \in K^{\chi^f}_{+,i} \text{ if } k<0 \end{cases} \ . \tag{V.33}$$

Under a spatial translation around the circle, the fields have the holonomy (V.19), that we infer from the relations

$$e^{-ik\ell} = e^{i\chi^f_{\alpha,i}} , \qquad \text{for} \qquad k \in K^{\chi^f}_{\alpha,i} . \tag{V.34}$$

As a consequence of the CAR for the creation and annihilation operators, the fermionic coordinates satisfy the CAR

$$\{\xi^\chi_{\alpha,i}(k)^\#, \xi^\chi_{\alpha',i'}(k')^{\#'}\} = \delta_{\alpha\alpha'}\delta_{ii'}\delta_{kk'}\delta_{\#^*\#'} I , \qquad \text{for} \qquad k \in K^{\chi^f}_{\alpha,i} , \ \ k' \in K^{\chi^f}_{\alpha',i'} . \tag{V.35}$$

Here $\delta_{\#^*\#'} = 0$ when $\#$ and $\#'$ are the same, while $\delta_{\#^*\#'} = 1$ when $\#$ and $\#'$ differ. Thus the above relations are shorthand for the relations $\{\xi^\chi_{\alpha,i}(k), \xi^\chi_{\alpha',i'}(k')\} = 0$, $\{\xi^\chi_{\alpha,i}(k), \xi^\chi_{\alpha',i'}(k')^*\} = \delta_{\alpha\alpha'}\delta_{ii'}\delta_{kk'} I$, and their adjoints. We infer that the fields satisfy the CAR

$$\{\psi^\chi_{\alpha,i}(x)^\#, \psi^\chi_{\alpha',i'}(x')^{\#'}\} = \delta_{\alpha\alpha'}\delta_{ii'}\delta_{\#^*\#'}\delta(x-x') I . \tag{V.36}$$

Here we use the representation for the Dirac measure with period ℓ, namely

$$\delta(x) = \frac{1}{\ell} \sum_{k \in K^{\chi^f}_{\alpha,i}} e^{-ikx} , \tag{V.37}$$

justified as in (II.29). We combine this with the calculation

$$\begin{aligned} &\left\{\psi^\chi_{\alpha,i}(x)^\#, \psi^\chi_{\alpha',i'}(x')^{\#'}\right\} \\ &\quad = \frac{1}{\ell} \sum_{k \in K^{\chi^f}_{\alpha,i}} \sum_{k' \in K^{\chi^f}_{\alpha',i'}} \left\{\xi^{\chi^f}_{\alpha,i}(k)^\#, \xi^{\chi^f}_{\alpha',i'}(k')^{\#'}\right\} e^{-ikx+ik'x'} \\ &\quad = \frac{1}{\ell} \sum_{k \in K^{\chi^f}_{\alpha,i}} \sum_{k' \in K^{\chi^f}_{\alpha',i'}} \delta_{\alpha\alpha'}\delta_{ii'}\delta_{kk'}\delta_{\#^*\#'} \, e^{-ik(x-x')} I = \delta_{\alpha\alpha'}\delta_{ii'}\delta_{\#^*\#'} \ \delta(x-x') I . \end{aligned} \tag{V.38}$$

Standard normal ordering of creation and annihilation operators is

$$\begin{aligned} :b^{\chi^f}_{\pm,i}(k) b^{\chi^f}_{\pm',i'}(k'): &= b^{\chi^f}_{\pm,i}(k) b^{\chi^f}_{\pm',i'}(k') , \qquad :b^{\chi^f}_{\pm,i}(k)^* b^{\chi^f}_{\pm',i'}(k')^*: = b^{\chi^f}_{\pm,i}(k)^* b^{\chi^f}_{\pm',i'}(k')^* , \\ :b^{\chi^f}_{\pm,i}(k)^* b^{\chi^f}_{\pm',i'}(k'): &= b^{\chi^f}_{\pm,i}(k)^* b^{\chi^f}_{\pm',i'}(k') , \qquad :b^{\chi^f}_{\pm,i}(k) b^{\chi^f}_{\pm',i'}(k')^*: = -b^{\chi^f}_{\pm',i'}(k')^* b^{\chi^f}_{\pm,i}(k) , \end{aligned} \tag{V.39}$$

extending linearly to the products of fields. The Hamiltonian for the free Dirac field is

$$H_0^{f,\chi} = \sum_{i=1}^n \left(\sum_{k \in K^{\chi^f}_{+,i}} |k| b^{\chi^f}_{+,i}(k)^* b^{\chi^f}_{+,i}(k) + \sum_{k \in K^{\chi^f}_{-,i}} |k| b^{\chi^f}_{-,i}(-k)^* b^{\chi^f}_{-,i}(-k) \right) , \tag{V.40}$$

and the momentum operator is

$$P^{f,\chi} = \sum_{i=1}^n \left(\sum_{k \in K^{\chi^f}_{+,i}} k b^{\chi^f}_{+,i}(k)^* b^{\chi^f}_{+,i}(k) - \sum_{k \in K^{\chi^f}_{-,i}} k b^{\chi^f}_{-,i}(-k)^* b^{\chi^f}_{-,i}(-k) \right) . \tag{V.41}$$

We can express the Hamiltonian and momentum operators in terms of the fields by

$$H_0^{f,\chi} + P^{f,\chi} = -2i \sum_{i=1}^{n} \int_0^{\ell} :\psi_{1,i}^{\chi}{}^* \, \partial_x \psi_{1,i}^{\chi}: \, dx \ , \qquad H_0^{f,\chi} - P^{f,\chi} = 2i \sum_{i=1}^{n} \int_0^{\ell} :\psi_{2,i}^{\chi}{}^* \, \partial_x \psi_{2,i}^{\chi}: \, dx \ , \quad \text{(V.42)}$$

where $\partial_x = \frac{\partial}{\partial x}$. Noting (V.14) and (V.16), we infer that (V.40) and (V.41) also equal

$$H_0^{f,\chi} = -i \int_0^{\ell} :\overline{\psi^{\chi}} \gamma^1 \partial_x \psi^{\chi}: \, dx \ , \qquad \text{and} \qquad P_0^{f,\chi} = -i \int_0^{\ell} :\overline{\psi^{\chi}} \, \gamma^0 \partial_x \psi^{\chi}: \, dx \ . \quad \text{(V.43)}$$

The real-time free field, with initial data equal to (V.30)–(V.31), is

$$\psi_{\mathrm{RT},\alpha,i}^{\chi}(x,t) = e^{itH_0^{f,\chi}} \psi_{\alpha,i}^{\chi}(x) e^{-itH_0^{f,\chi}} = e^{itH_0^{f,\chi} - ixP^{f,\chi}} \psi_{\alpha,i}^{\chi}(0) e^{-itH_0^{f,\chi} + ixP^{f,\chi}} \ . \quad \text{(V.44)}$$

V.2 The Real-Time Dirac Equation

Define the real-time Dirac operator as

$$\partial\!\!\!/ = \gamma^0 \frac{\partial}{\partial t} + \gamma^1 \frac{\partial}{\partial x} = \begin{pmatrix} 0 & -i\frac{\partial}{\partial t} + i\frac{\partial}{\partial x} \\ i\frac{\partial}{\partial t} + i\frac{\partial}{\partial x} & 0 \end{pmatrix} \ . \quad \text{(V.45)}$$

This operator is neither symmetric nor skew-symmetric. The corresponding real-time Dirac equation is

$$i\partial\!\!\!/ \, \psi_{\mathrm{RT},j}^{\chi}(x,t) = 0 \ , \quad \text{(V.46)}$$

where the factor i is conventional. In terms of components, one can write the equations for left-moving and for right-moving solutions respectively as,

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) \psi_{\mathrm{RT},2,j}^{\chi}(x,t) = 0 \ , \qquad \text{and} \qquad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right) \psi_{\mathrm{RT},1,j}^{\chi}(x,t) = 0 \ . \quad \text{(V.47)}$$

We mention the *charge conjugation* transformation

$$\psi_{\mathrm{RT}}^{\chi}(x,t) \to \psi_{\mathrm{RT}}^{\chi}(x,t)^{\mathrm{c}} = C\overline{\psi_{\mathrm{RT}}^{\chi}}(x,t)^{\mathrm{T}} \quad \text{(V.48)}$$

for the Dirac field. We retain the previous definition (V.8)–(V.9) adapted to the situation with n-copies of the two-component, real-time field, namely

$$\psi_{\mathrm{RT}}^{\chi}(x,t) = \begin{pmatrix} \psi_{\mathrm{RT},1,1}^{\chi}(x,t) \\ \psi_{\mathrm{RT},2,1}^{\chi}(x,t) \\ \psi_{\mathrm{RT},1,2}^{\chi}(x,t) \\ \vdots \\ \psi_{\mathrm{RT},2,n}^{\chi}(x,t) \end{pmatrix} \ , \qquad \text{and} \qquad \psi_{\mathrm{RT}}^{\chi}(x,t)^{\mathrm{c}} = \begin{pmatrix} \psi_{\mathrm{RT},1,1}^{\chi}(x,t)^* \\ \psi_{\mathrm{RT},2,1}^{\chi}(x,t)^* \\ \psi_{\mathrm{RT},1,2}^{\chi}(x,t)^* \\ \vdots \\ \psi_{\mathrm{RT},2,n}^{\chi}(x,t)^* \end{pmatrix} \ . \quad \text{(V.49)}$$

Here $\psi^{\chi}_{\mathrm{RT},\alpha,i}(x,t)^*$ denotes the hermitian conjugate of $\psi^{\chi}_{\mathrm{RT},\alpha,i}(x,t)$. The condition that $\psi^{\chi}_{\mathrm{RT}}$ is charge self-conjugate then reduces to $\psi^{\chi}_{\mathrm{RT}}$ being a real (Majorana) spinor. For the field (V.49) with n copies, we also use $\partial\!\!\!/$ to denote the real-time Dirac operator acting on each copy, and the matrices γ^μ and C also act as block matrices composed of n identical copies. Charge conjugation maps a solution $\psi^{\chi}_{\mathrm{RT}}(x,t)$ to the Dirac equation into the charge conjugate solution $\psi^{\chi}_{\mathrm{RT}}(x,t)^{\mathrm{c}}$. To derive this, take the complex conjugate of (V.46) (writing $\overline{\gamma^\mu}$ the matrix complex conjugate to γ^μ) and multiply by $C\,(\gamma^0)^{\mathrm{T}}$. We obtain the Dirac equation for a charge conjugate as long as

$$C\left(\gamma^0\right)^{\mathrm{T}}\,\overline{\gamma^\mu} = \text{const.}\,\gamma^\mu\,C\left(\gamma^0\right)^{\mathrm{T}}\;. \tag{V.50}$$

In our purely imaginary representation of the Dirac matrices, $\overline{\gamma^\mu} = -\gamma^\mu$, so our earlier choice $C\,(\gamma^0)^{\mathrm{T}} = I$ in (V.10) yields the constant in (V.50) equal to -1.

In the following section we also study the massive Dirac equation. With our choice $C\,(\gamma^0)^{\mathrm{T}} = I$, the charge conjugation transformation also maps solutions of the massive Dirac equation $\left(i\partial\!\!\!/ - m\right)\psi_{\mathrm{RT}}(x,t) = 0$ into (charge-conjugate) solutions that satisfy $\left(i\partial\!\!\!/ - m\right)\psi_{\mathrm{RT}}(x,t)^{\mathrm{c}} = 0$.

V.3 Twist Symmetry

We introduce a self-adjoint twist generator $J^{f,\chi}$. This operator acts on the fermionic Hilbert space and generates a unitary group $U^{f,\chi}(\theta) = e^{i\theta J^{f,\chi}}$ that twists each component $\psi^{\chi}_{\alpha,i}$-component of the Dirac field. The twist is by a phase $e^{i\Omega^f_{\alpha,i}\theta}$, where the twisting angles are proportional to the set of independent, real twist parameters $\{\Omega^f_{\alpha,i}\}$ that we specify. In other words,

$$U^{f,\chi}(\theta)\psi^{\chi}_{\mathrm{RT},\alpha,i}U^{f,\chi}(\theta)^* = e^{i\Omega^f_{\alpha,i}\theta}\,\psi^{\chi}_{\mathrm{RT}\alpha,i}\;. \tag{V.51}$$

Define the fermionic twist generator as

$$J^{f,\chi} = \sum_{i=1}^{n}\sum_{\alpha=1}^{2}\Omega^f_{\alpha,i}\int_0^\ell :\psi^{\chi}_{\alpha,i}(x)\,\psi^{\chi}_{\alpha,i}(x)^*:dx + M(\Omega)\;, \tag{V.52}$$

where $M(\Omega)$ is the constant

$$M(\Omega) = \frac{1}{2}\sum_{i=1}^{n}\left(\Omega^f_{1,i} - \Omega^f_{2,i}\right)\;. \tag{V.53}$$

In the bosonic case, we chose the additive constant in the bosonic twist generator so that $J^{b,\chi}$ annihilates the ground state of the bosonic Hamiltonian H^b; this led to the twist positivity property of the bosonic partition function. Fermionic partition functions do not have the twist positivity property, so we now explain the rational for our choice of $M(\Omega)$. An elementary computation shows that

$$\int_0^\ell :\psi^{\chi}_{1,i}(x)\,\psi^{\chi}_{1,i}(x)^*:dx = \sum_{\substack{k>0\\ k\in K^{\chi^f}_{+,i}}} b^{\chi^f}_{+,i}(k)^* b^{\chi^f}_{+,i}(k) - \sum_{\substack{k<0\\ k\in K^{\chi^f}_{-,i}}} b^{\chi^f}_{-,i}(-k)^* b^{\chi^f}_{-,i}(-k)\;, \tag{V.54}$$

and

$$\int_0^\ell :\psi^\chi_{2,i}(x)\,\psi^\chi_{2,i}(x)^*:dx = \sum_{\substack{k<0\\ k\in K^{\chi^f}_{+,i}}} b^{\chi^f}_{+,i}(k)^* b^{\chi^f}_{+,i}(k) - \sum_{\substack{k>0\\ k\in K^{\chi^f}_{-,i}}} b^{\chi^f}_{-,i}(-k)^* b^{\chi^f}_{-,i}(-k)\ . \tag{V.55}$$

Thus $J^{f,\chi}$ is a sum of commuting, self-adjoint generators for each component of the fermionic fields, and has a simple expression in terms of the positive and negative frequency parts of the number operators. Let us define angles for the twist generation for the creation operators,

$$\Omega^f_i(k) = \begin{cases} \Omega^f_{1,i}\ , & \text{if } k>0 \\ \Omega^f_{2,i}\ , & \text{if } k<0 \end{cases}\ . \tag{V.56}$$

Then we can write $J^{f,\chi}$ as

$$J^{f,\chi} = \sum_{i=1}^n \sum_{k\in K^{\chi^f}_{+,i}} \Omega^f_i(k)\, N^{f,\chi}_{+,i} - \sum_{i=1}^n \sum_{k\in K^{\chi^f}_{-,i}} \Omega^f_i(-k)\, N^{f,\chi}_{-,i} + M(\Omega)\ . \tag{V.57}$$

Next we pair each momentum $k \in K^{\chi^f}_{+,i}$ with a dual momentum $\tilde{k} \in K^{\chi^f}_{-,i}$ such that

$$k + \tilde{k} = -\chi^f_{1,i} - \chi^f_{2,i}\ . \tag{V.58}$$

Note that k and $\tilde{k}$ have opposite signs, unless $\ell k = -\chi^f_{\pm,i}$, namely unless k is the momentum in $K^{\chi^f}_{+,i}$ or in $K^{\chi^f}_{-,i}$ that is closest to zero. In particular, if also we take $k>0$, then $\tilde{k}<0$. Accounting for these relationships, we rewrite the fermionic twist generator $J^{f,\chi}$ in the form

$$\begin{aligned} J^{f,\chi} \;=\; & \sum_{i=1}^n \sum_{\substack{k>0\\ k\in K^{\chi^f}_{+,i}}} \left(\Omega^f_i(k)\left(N^{f,\chi}_{+,i}(k) - \frac{1}{2}\right) - \Omega^f_i(-\tilde{k})\left(N^{f,\chi}_{-,i}(-\tilde{k}) - \frac{1}{2}\right)\right. \\ & \left. + \Omega^f_i(\tilde{k})\left(N^{f,\chi}_{+,i}(\tilde{k}) - \frac{1}{2}\right) - \Omega^f_i(-k)\left(N^{f,\chi}_{-,i}(-k) - \frac{1}{2}\right)\right) \\ & - \sum_{i=1}^n \left(\Omega^f_i(\frac{\chi^f_{1,i}}{\ell})\left(N^{f,\chi}_{-,i}(-\chi^f_{1,i}) - \frac{1}{2}\right) - \Omega^f_i(-\frac{\chi^f_{2,i}}{\ell})\left(N^{f,\chi}_{+,i}(\chi^f_{2,i}) - \frac{1}{2}\right)\right)\ . \end{aligned} \tag{V.59}$$

In the representation (V.59), the four factors factors of $\frac{1}{2}$ in the summand over $k>0$ actually cancel identically. On the other hand, the remaining two factors of $\frac{1}{2}$ that occur outside the sum over k reflect the constant

$$M(\Omega) = \sum_{i=1}^n \left(\frac{1}{2}\Omega^f_i(\frac{\chi^f_{1,i}}{\ell}) - \frac{1}{2}\Omega^f_i(-\frac{\chi^f_{2,i}}{\ell})\right) = \frac{1}{2}\sum_{i=1}^n \left(\Omega^f_{1,i} - \Omega^f_{2,i}\right)\ , \tag{V.60}$$

as chosen in (V.53). The representation (V.59) exhibits that the operators $\pm J^{f,\chi}$ have the same spectrum and the same spectral multiplicities. In fact, each summand in this representation has this property. This justifies our choice of the constant $M(\Omega)$ that normalizes $U^{f,\chi}(\theta)$.

Since the twist generator $J^{f,\chi}$ is an elementary function of the number operators, it commutes with the free fermionic Hamiltonian and with the momentum operator,

$$[J^{f,\chi}, H_0^{f,\chi}] = 0 \;, \qquad \text{and} \qquad [J^{f,\chi}, P^{f,\chi}] = 0 \;. \tag{V.61}$$

Hence the two-parameter group

$$U^{f,\chi}(\theta,\sigma) = U^{f,\chi}(\theta) e^{i\sigma P^{f,\chi}} \tag{V.62}$$

is an abelian symmetry group of both $H_0^{f,\chi}$ and $P^{f,\chi}$,

$$U^{f,\chi}(\theta,\sigma)\, H_0^{f,\chi} = H_0^{f,\chi}\, U^{f,\chi}(\theta,\sigma) \qquad \text{and} \qquad U^{f,\chi}(\theta,\sigma)\, P_0^{f,\chi} = P_0^{f,\chi}\, U^{f,\chi}(\theta,\sigma) \;. \tag{V.63}$$

We can summarize the above symmetries by the twist relation for the real-time field

$$U^{f,\chi}(\theta,\sigma)\, \psi^{\chi}_{\mathrm{RT},\alpha,i}(x+\ell,t)\, U^{f,\chi}(\theta,\sigma)^* = e^{i\theta\Omega^f_{\alpha,i} + i\chi^f_{\alpha,i}}\; \psi^{\chi}_{\mathrm{RT},\alpha,i}(x-\sigma,t) \;, \tag{V.64}$$

where $\theta, \sigma \in \mathbb{R}$.

V.4 Partition Functions

Define the free fermionic partition function $\mathfrak{Z}^f(\mathcal{T})$ by

$$\mathfrak{Z}^f\,(\mathcal{T}) = \mathrm{Tr}_{\mathcal{H}^{f,\chi}} \left(\Gamma U^{f,\chi}(\sigma,\theta)^* e^{-\beta H_0^{f,\chi}} \right) = \mathrm{Tr}_{\mathcal{H}^{f,\chi}} \left(\Gamma e^{-i\sigma P^{f,\chi} - i\theta J^{f,\chi} - \beta H_0^{f,\chi}} \right) \;. \tag{V.65}$$

Here $\Gamma = (-I)^{N^{f,\chi}}$ is the $\mathbb{Z}_2$-grading defined by the total fermion number operator $N^{f,\chi}$,

$$N^{f,\chi} = \sum_{i=1}^{n} \sum_{k\in K^{\chi^f}_{+,i}} b^{\chi^f}_{+,i}(k)^* b^{\chi^f}_{+,i}(k) + \sum_{i=1}^{n} \sum_{k\in K^{\chi^f}_{-,i}} b^{\chi^f}_{-,i}(-k)^* b^{\chi^f}_{-,i}(-k) \;. \tag{V.66}$$

The operators Γ, $U^{f,\chi}(\sigma,\theta)$, and $H_0^{f,\chi}$ not only mutually commute, but they all have simultaneous eigenstates in $\mathcal{H}^{f,\chi}$ labelled by the states with $n_{\pm,i,k} = 0$ or 1 quanta created by $b^{\chi^f}_{\pm,i}(k)^*$.

In terms of these parameters, let

$$\gamma^f_{+,i}(k) = e^{-i\theta\Omega^f_i(k) - i\sigma k - \beta|k|} \;,\; \text{ with } k \in K^{\chi^f}_{+,i} \;,\; \text{ and } \gamma^f_{-,i}(-k) = e^{-i\theta\Omega^f_i(-k) - i\sigma k - \beta|k|} \;.\; \text{with } k \in K^{\chi^f}_{-,i} \tag{V.67}$$

Also let

$$\gamma^f_{\alpha,i}(k) = e^{-i\theta\Omega^f_{\alpha,i} - i\sigma k - \beta|k|} \;,\; \text{ with } k \in K^{\chi^f}_{\alpha,i} \;,\; \text{ for } \alpha = 1,2 \;. \tag{V.68}$$

Since we assume that the fermionic twists satisfy (V.18), we infer that allowed momenta satisfy $k \neq 0$, and consequently $\gamma^f_{\alpha,i}(k) \neq 1$. We then establish as in the bosonic case:

Proposition V.4.1. *Let $\beta > 0$, and assume the fermionic twists satisfy the non-triviality condition (V.18).*

a. *The partition function is given by the convergent product*

$$\mathfrak{Z}^f(\mathcal{T}) = e^{-i\sum_{i=1}^{n}(\Omega^f_{1,i}-\Omega^f_{2,i})/2}\prod_{i=i}^{n}\left(\prod_{k\in K^{\chi^f}_{+,i}}(1-\gamma_{+,i}(k))\prod_{k'\in K^{\chi^f}_{-,i}}(1-\gamma_{-,i}(-k'))\right)\neq 0\ . \qquad \text{(V.69)}$$

b. *If $\chi^f_{1,i} = \chi^f_{2,i}$ and $\Omega^f_{1,i} = \Omega^f_{2,i}$ for all i, then $\mathfrak{Z}^f(\mathcal{T})$ is positive.*

c. *If also $\chi^f_{\alpha,i} = \chi^b_i$ and $\Omega^f_{\alpha,i} = \Omega^b_i$ for all α, i, then $\mathfrak{Z}^f(\mathcal{T})$ and $\mathfrak{Z}^b(\mathcal{T})$ given by (II.45) are inverses of one another.*

V.5 Imaginary-Time Dirac Fields and Pair Correlations

In this section, we define the *imaginary time* Dirac field $\psi^\chi(x,t)$. We also define the fermionic expectation $\langle\ \cdot\ \rangle^{f,\chi}_{\mathcal{T}}$, analagous to the Bosonic expectations of §II.3.

Define the imaginary-time fermionic field by

$$\psi^\chi(x,t) = \psi^\chi_{\mathrm{RT}}(x,it) = e^{-tH_0^{f,\chi}}\psi^f(x)e^{tH_0^{f,\chi}}\ . \qquad \text{(V.70)}$$

Use the adjoint $\overline{\psi} = \psi^*\gamma^0$, and define

$$\left(\overline{\psi^\chi}\right)(x,t) = \left(\overline{\psi^\chi_{\mathrm{RT}}}\right)(x,it) = e^{-tH_0^{f,\chi}}\overline{\psi^\chi}(x)e^{tH_0^{f,\chi}}\ , \qquad \text{(V.71)}$$

with components $\left(\overline{\psi^\chi}\right)_{\alpha,i}(x,t)$. To simplify notation we write the components of the adjoint field as

$$\overline{\psi^\chi_{\alpha,i}}(x,t) = \left(\overline{\psi^\chi}\right)_{\alpha,i}(x,t)\ . \qquad \text{(V.72)}$$

Now we define the expectation. Use the unitary element $\Gamma U^{f,\chi}(\sigma,\theta)^*$ to twist expectations, and regularize the expectation by the fermionic heat kernel $e^{-\beta H_0^{f,\chi}}$, which is trace class. Denote the parameters for the Dirac field by

$$\mathcal{T} = \{\chi, \theta\Omega, \sigma, \ell, \beta\}\ , \qquad \text{(V.73)}$$

In Proposition V.4.1 we saw that the fermionic partition function does not vanish, $\mathfrak{Z}^f(\mathcal{T}) \neq 0$, so one can normalize the fermionic expectation,

$$\langle\ \cdot\ \rangle^{f,\chi}_{\mathcal{T}} = \frac{\mathrm{Tr}_{\mathcal{H}^{f,\chi}}\left(\ \cdot\ \Gamma e^{-i\theta J^{f,\chi}-i\sigma P^{f,\chi}-\beta H_0^{f,\chi}}\right)}{\mathrm{Tr}_{\mathcal{H}^{f,\chi}}\left(\Gamma e^{-i\theta J^{f,\chi}-i\sigma P^{f,\chi}-\beta H_0^{f,\chi}}\right)}\ . \qquad \text{(V.74)}$$

The normalized expectation has the property that $\langle I\rangle_{\mathcal{T}}^{f,\chi} = 1$. Furthermore, the expectation is twist-invariant, in the sense that for an operator T for which the expectation is defined,

$$\langle T\rangle_{\mathcal{T}}^{f,\chi} = \left\langle U^{f,\chi}(\theta)\, T\, U^{f,\chi}(\theta)^* \right\rangle_{\mathcal{T}}^{f,\chi} \;. \tag{V.75}$$

As the time-zero fermion fields have a non-zero spatial twist, namely as given by (V.51), and the imaginary time fields have the same twist transformation law,

$$U^{f,\chi}(\theta)\, \psi^{\chi}_{\alpha,i}(x,t)\, U^{f,\chi}(\theta)^* = e^{i\theta\Omega^f_{\alpha,i}}\; \psi^{\chi}_{\alpha,i}(x,t) \;, \tag{V.76}$$

it follows that

$$\left\langle \psi^{\chi}_{\alpha,i}(x,t)\right\rangle_{\mathcal{T}}^{f,\chi} = \left\langle \overline{\psi^{\chi}_{\alpha,i}}(x,t)\right\rangle_{\mathcal{T}}^{f,\chi} = 0 \;. \tag{V.77}$$

Similarly, for $0 \le t \le t' \le \beta$,

$$\left\langle \psi^{\chi}_{\alpha,i}(x,t)\, \psi^{\chi}_{\alpha',j}(x',t')\right\rangle_{\mathcal{T}}^{f,\chi} = \left\langle \overline{\psi^{\chi}_{\alpha,i}}(x,t)\, \overline{\psi^{\chi}_{\alpha',j}}(x',t')\right\rangle_{\mathcal{T}}^{f,\chi} = 0 \;. \tag{V.78}$$

Define the time-ordered product of two components of the fermion field as

$$\left(\overline{\psi^{\chi}_{\alpha,i}}(x,t)\psi^{\chi}_{\alpha',j}(x',t')\right)_+ = \begin{cases} \overline{\psi^{\chi}_{\alpha,i}}(x,t)\, \psi^{\chi}_{\alpha',j}(x',t') \;, & \text{if } t < t' \\ -\psi^{\chi}_{\alpha',j}(x',t')\, \overline{\psi^{\chi}_{\alpha,i}}(x,t) \;, & \text{if } t' < t \end{cases} \;. \tag{V.79}$$

We have not defined the time-ordered product for $t = t'$, and the two limits $\lim_{t\to t\pm}$ of the time ordered product (V.79) differ on the diagonal $(x,t) = (x',t')$. The fermionic pair correlation matrix $S^{\chi}(x-x',t-t')$ is the time-ordered expectation of two Dirac fields,

$$S^{\chi}_{\mathcal{T},\alpha\alpha',ij}(x-x',t-t') = \left\langle \left(\overline{\psi^{\chi}}_{\alpha,i}(x,t)\, \psi^{\chi}_{\alpha',j}(x',t')\right)_+ \right\rangle_{\mathcal{T}}^{f,\chi} \;. \tag{V.80}$$

Osterwalder and Schrader proved [11], as part of their construction of quantum fields from Euclidean Green's functions, that the above ambiguity of the pair correlation matrix (V.80) on the diagonal $(x,t) = (x',t')$ does not affect the real-time quantum field theory.

Furthermore, the matrix elements vanish for $i \neq j$, so we denote the $i = j$ entries with one fewer indices,

$$S^{\chi}_{\mathcal{T},\alpha\alpha',ij}(x-x',t-t') = \delta_{ij} S^{\chi}_{\mathcal{T},\alpha\alpha',i}(x-x',t-t') \;. \tag{V.81}$$

It is natural to introduce the matrices $S^{\chi}_{\mathcal{T},i}(x-x',t-t')$ of 2×2 blocks for each i, with entries $S^{\chi}_{\mathcal{T},\alpha\alpha',i}(x-x',t-t')$. Let

$$\begin{aligned} S^{\chi}_{\mathcal{T},i}(x-x',t-t') &= \begin{pmatrix} S^{\chi}_{\mathcal{T},11,i}(x-x',t-t') & S^{\chi}_{\mathcal{T},12,i}(x-x',t-t') \\ S^{\chi}_{\mathcal{T},21,i}(x-x',t-t') & S^{\chi}_{\mathcal{T},22,i}(x-x',t-t') \end{pmatrix} \\ &= \begin{pmatrix} 0 & i\left\langle \left(\psi^{\chi}_{2,i}(x,t)^*\, \psi^{\chi}_{2,i}(x',t')\right)_+ \right\rangle_{\mathcal{T}}^{f,\chi} \\ -i\left\langle \left(\psi^{\chi}_{1,i}(x,t)^*\, \psi^{\chi}_{1,i}(x',t')\right)_+ \right\rangle_{\mathcal{T}}^{f,\chi} & 0 \end{pmatrix} . \end{aligned} \tag{V.82}$$

V.6 Fermion Holonomy Moves

We explain the idea of *holonomy moves*, a useful set of identities to evaluate expectations. We introduced this method in [8] and elaborated in the bosonic case in [2]. Here we give the corresponding elaboration relevant for the expectation of fermion operators. We define an opeartor X to be fermionic, if $\Gamma X \Gamma = -X$. Let

$$\mathfrak{K} = \mathfrak{K}(\sigma, \theta, \beta) = \Gamma\, U^{f,\chi}(\sigma,\theta)^* \, e^{-\beta H_0^{f,\chi}} = \Gamma e^{-i\theta J^{f,\chi} - i\sigma P^{f,\chi} - \beta H_0^{f,\chi}} \ . \tag{V.83}$$

We say the operator X has an elementary holonomy law with respect to the expectation $\langle \cdot \rangle_{\mathcal{T}}^{f,\chi}$ defined in (V.74), if

$$X\mathfrak{K} = \pm s \mathfrak{K} X \ , \qquad \text{where} \quad s \in \mathbb{C} \ , \ \text{and} \ s \neq 1. \tag{V.84}$$

We call this a *bosonic* holonomy law in the case of the plus sign, and we call it a *fermionic* holonomy law in case of the minus sign.

Proposition V.6.1. *Let X denote an operator that has a non-trivial, elementary holonomy law with respect to the expectation $\langle \cdot \rangle_{\mathcal{T}}^{f,\chi}$. Then*

$$\langle XY \rangle_{\mathcal{T}}^{f,\chi} = \frac{1}{(1-s)} \langle \{X, Y\} \rangle_{\mathcal{T}}^{f,\chi} = \frac{-s^{-1}}{(1-s^{-1})} \langle \{X, Y\} \rangle_{\mathcal{T}}^{f,\chi} \ . \tag{V.85}$$

Remark. We call the identity (V.85) a *fermionic holonomy move.*

Proof. It is sufficient to prove the first identity. Take the expectation of the identity $XY = -YX + \{X, Y\}$, to obtain $\langle XY \rangle_{\mathcal{T}}^{f,\chi} = -\langle YX \rangle_{\mathcal{T}}^{f,\chi} + \langle \{X, Y\} \rangle_{\mathcal{T}}^{f,\chi}$. Using the definition of the expectation (V.74), the elementary holonomy move identity, and cyclicity of the trace, we infer

$$\langle XY \rangle_{\mathcal{T}}^{f,\chi} = s \langle XY \rangle_{\mathcal{T}}^{f,\chi} + \langle \{X, Y\} \rangle_{\mathcal{T}}^{f,\chi} \ , \tag{V.86}$$

yielding (V.85).

V.7 Evaluation of the Pair Correlation Matrix

We use the fermionic holonomy identity (V.85), among other things, to compute the pair correlation matrix (V.82). For real $u \neq 0$, define the step function

$$\theta(u) = \begin{cases} 1 \ , & \text{if } u > 0 \\ 0 \ , & \text{if } u < 0 \end{cases} \ . \tag{V.87}$$

Using the definition of the fields we obtain

Proposition V.7.1. *For $0 \leq t, t' \leq \beta$, and $0 < |t - t'| < \beta$, the non-zero elements of the pair correlation matrix (V.82) are given by absolutely convergent Fourier representations*

$$\begin{aligned} & S^{\chi}_{21,j}(x - x', t - t') \\ &= -i \left\langle \left(\psi^{\chi}_{1,j}(x,t)^* \psi^{\chi}_{1,j}(x',t') \right)_+ \right\rangle^{f,\chi}_{\mathcal{T}} \\ &= \frac{i}{\ell} \sum_{k \in K^{\chi^f}_{1,j}} \left\{ \theta(-k) \left(\frac{\overline{\gamma^f_{1,j}(k)}}{1 - \overline{\gamma^f_{1,j}(k)}} + \theta(t - t') \right) e^{-|k|(t-t')} - \theta(k) \left(\frac{\gamma^f_{1,j}(k)}{1 - \gamma^f_{1,j}(k)} + \theta(t' - t) \right) e^{|k|(t-t')} \right\} e^{ik(x-x')} , \end{aligned} \tag{V.88}$$

and

$$\begin{aligned} & S^{\chi}_{12,j}(x - x', t - t') \\ &= i \left\langle \left(\psi^{\chi}_{2,j}(x,t)^* \psi^{\chi}_{2,j}(x',t') \right)_+ \right\rangle^{f,\chi}_{\mathcal{T}} \\ &= \frac{i}{\ell} \sum_{k \in K^{\chi^f}_{2,j}} \left\{ \theta(-k) \left(\frac{\gamma^f_{2,j}(k)}{1 - \gamma^f_{2,j}(k)} + \theta(t' - t) \right) e^{|k|(t-t')} - \theta(k) \left(\frac{\overline{\gamma^f_{2,j}(k)}}{1 - \overline{\gamma^f_{2,j}(k)}} + \theta(t - t') \right) e^{-|k|(t-t')} \right\} e^{ik(x-x')} . \end{aligned} \tag{V.89}$$

Furthermore if $\Omega^f_{1,j} = \Omega^f_{2,j}$ and $\chi^f_{1,j} = \chi^f_{2,j}$, then S^{χ} satisfies the hermiticity condition

$$S^{\chi}_{12,j}(x - x', t - t') = \overline{S^{\chi}_{21,j}(x' - x, t' - t)} . \tag{V.90}$$

Proof. First assume the above representations, and consider their convergence. By assumption, $0 < |t' - t| < \beta$. Therefore the terms in (V.88)–(V.89) that are proportional to $\theta(\pm(t - t'))$ are well-defined, and the magnitude of these terms decay exponentially as $|k| \to \infty$. This remark also ensures that terms in the sums (V.88)–(V.89) proportional to the exponentially growing functions $e^{|k||t-t'|}$ always occur multiplied by either $\gamma^f_{\alpha,j}(k)$ or $\overline{\gamma^f_{\alpha,j}(k)}$. Hence the bound $|\gamma^f_{\alpha,j}(k)| \leq e^{-|k|\beta}$ ensures that these terms also converge to zero exponentially as $|k| \to \infty$. Therefore each sum converges absolutely.

We now compute $\left\langle \left(\psi^{\chi}_{1,j}(x,t)^* \psi^{\chi}_{1,j}(x',t') \right)_+ \right\rangle^{f,\chi}_{\mathcal{T}}$ for $0 \leq t \leq t' \leq \beta$. In this case, use cyclicity and the diagonal nature of the trace to obtain

$$\begin{aligned} \left\langle \left(\psi^{\chi}_{1,j}(x,t)^* \psi^{\chi}_{1,j}(x',t') \right)_+ \right\rangle^{f,\chi}_{\mathcal{T}} &= \frac{1}{\ell} \sum_{\substack{k>0 \\ k \in K^{\chi^f}_{1,j}}} \left\langle b^{\chi^f}_{+,j}(k) b^{\chi^f}_{+,j}(k)^*(t' - t) \right\rangle^{f,\chi}_{\mathcal{T}} e^{ik(x-x')} \\ &\quad + \frac{1}{\ell} \sum_{\substack{k<0 \\ k \in K^{\chi^f}_{1,j}}} \left\langle b^{\chi^f}_{-,j}(-k) b^{\chi^f}_{-,j}(-k)^*(t' - t) \right\rangle^{f,\chi}_{\mathcal{T}} e^{ik(x-x')} . \end{aligned} \tag{V.91}$$

The operators $b^{\chi^f}_{\pm,j}(\pm k)$ have a fermionic holonomy law with respect to the expectation in question. In particular,

$$b^{\chi^f}_{+,j}(k)\,\mathfrak{K} = -\gamma^f_{1,j}(k)\,\mathfrak{K}\,b^{\chi^f}_{+,j}(k)\;, \qquad \text{and} \qquad b^{\chi^f}_{-,j}(-k)^*\mathfrak{K} = -\left(\overline{\gamma^f_{1,j}(k)}\right)^{-1}\mathfrak{K}\,b^{\chi^f}_{-,j}(-k)^*\;. \tag{V.92}$$

Furthermore,

$$\{b^{\chi^f}_{+,j}(k), b^{\chi^f}_{+,j}(k)^*(t'-t)\} = \{b^{\chi^f}_{+,j}(k), b^{\chi^f}_{+,j}(k)^*\}e^{-\beta|k|(t'-t)} = e^{-\beta|k|(t'-t)}\;, \tag{V.93}$$

and

$$\{b^{\chi^f}_{-,j}(-k)^*, b^{\chi^f}_{-,j}(-k)(t'-t)\} = \{b^{\chi^f}_{-,j}(-k)^*, b^{\chi^f}_{-,j}(-k)\}e^{\beta|k|(t'-t)} = e^{\beta|k|(t'-t)}\;. \tag{V.94}$$

So from Proposition V.6.1 we infer (V.88). The computation for $0 \le t' \le t \le \beta$ and for the other component of S^χ_j are similar, so we do not include further details of the derivation of (V.88)–(V.89). Finally, we remark that the condition $\Omega^f_{1,j} = \Omega^f_{2,j}$ and $\chi^f_{1,j} = \chi^f_{2,j}$, ensures that $K^f_{1,j} = K^f_{2,j}$. Also for each $k \in K^f_{1,j}$, it follows that $\gamma^f_{1,j}(k) = \gamma^f_{2,j}(k)$. We then read off the symmetry relation (V.90) from (V.88)–(V.89).

V.8 The Euclidean Dirac Operator

In this section we define and study the Euclidean Dirac operator $\partial\!\!\!/_E$, and its symmetries, on a Hilbert space of square-integrable functions $\mathfrak{T}$. Functions in $\mathfrak{T} = \{f_{\alpha,i}(x,t)\}$ have $2n$ components, each an L^2 function on the space-time $\Sigma = S^1 \times [0,\beta]$. This takes into account the n copies, each with 2 components. Let $\mathfrak{T}_i$ denote the Hilbert space of L^2 functions for a single copy of the test function space,

$$\mathfrak{T}_i = L^2(\Sigma; dxdt) \oplus L^2(\Sigma; dxdt)\;. \tag{V.95}$$

Likewise let $\mathfrak{T}$ denote the direct sum over the n copies,

$$\mathfrak{T} = \oplus_{i=1}^n \mathfrak{T}_i\;. \tag{V.96}$$

Elements $f = \{f_{\alpha,i}\} \in C_0^\infty \cap \mathfrak{T}$ that are smooth and compactly supported are test functions for the Euclidean Dirac fields, and these functions pair with the imaginary time Dirac fields ψ^χ according to

$$\psi^\chi(f) = \sum_{\alpha,i} \psi^\chi_{\alpha,i}(f_{\alpha,i}) = \sum_{\alpha,i} \int_0^\ell dx \int_0^\beta dt\; \psi^\chi_{\alpha,i}(x,t) f_{\alpha,i}(x,t)\;. \tag{V.97}$$

We extend the domain of test functions below.

In this section, we also define the pair correlation operator $S^\chi_{\mathcal{T}}$. The operator $S^\chi_{\mathcal{T}}$ is the integral operator whose integral kernel is the pair correlation matrix (V.80). We will prove that $S^\chi_{\mathcal{T}}$ is a bounded operator on $\mathfrak{T}$, and that $S^\chi_{\mathcal{T}} = \left(\partial\!\!\!/_E\right)^{-1}$. In other words, the pair correlation matrix is the Green's funciton for the Euclidean Dirac operator.

The real Dirac matrices (V.3) are Hermitian and skew-Hermitian respectively. Define the Euclidean Dirac matrices by

$$\gamma_E^0 = -i\gamma^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \qquad \gamma_E^1 = \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} . \tag{V.98}$$

Using these matrices, both skew-Hermitian, define the Euclidean Dirac operator as

$$\partial\!\!\!/_E = \gamma_E^0 \frac{\partial}{\partial t} + \gamma_E^1 \frac{\partial}{\partial x} = \begin{pmatrix} 0 & -\frac{\partial}{\partial t} + i\frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} + i\frac{\partial}{\partial x} & 0 \end{pmatrix} . \tag{V.99}$$

Also let $\partial\!\!\!/_E$ denote the direct sum of n copies of this operator acting on the n-copies of the two-component Dirac field, or as an operator on the Hilbert $\mathfrak{T}$. Correspondingly let $\partial\!\!\!/_{E,i}$ denote the action of $\partial\!\!\!/_E$ restricted to $\mathfrak{T}_i$. This will be a diagonal 2×2 block of the form (V.99) in the matrix for $\partial\!\!\!/_E$ on the Hilbert space $\mathfrak{T}$, namely

$$\partial\!\!\!/_E = \begin{pmatrix} \partial\!\!\!/_{E,1} & 0 & \cdots & 0 \\ 0 & \partial\!\!\!/_{E,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \partial\!\!\!/_{E,n} \end{pmatrix} . \tag{V.100}$$

In order to study $\partial\!\!\!/_E$ as an operator on $\mathfrak{T}$, we must specify its domain. Begin with the domain C_0^∞ of smooth, $2n$-component, compactly-supported functions. Then each $\partial\!\!\!/_{E,i}$ is symmetric; for $f, g \in C_0^\infty$, we have

$$\left\langle \partial\!\!\!/_E f, g \right\rangle = \left\langle f, \partial\!\!\!/_E g \right\rangle . \tag{V.101}$$

But the operator $\partial\!\!\!/_E$ defined on C_0^∞ is not essentially self-adjoint. The elements of the defect spaces $\mathcal{D}_\pm$ for this operator are the spaces of square-integrable solutions f to the (mass $= 1$) Dirac equations

$$\left(\partial\!\!\!/_E \mp i \right) f = 0 . \tag{V.102}$$

Each defect space is infinite dimensional. For example, given j with $1 \le j \le n$, and any $k \in \mathbb{R}$, the vector

$$\begin{pmatrix} f_{1,j'} \\ f_{2,j'} \end{pmatrix} = \delta_{jj'} \begin{pmatrix} i \\ \sqrt{k^2+1} - k \end{pmatrix} e^{ikx + t\sqrt{k^2+1}} \tag{V.103}$$

is an element of $\mathcal{D}_+$.

In order to specify the operator $\partial\!\!\!/_{E,i}$ as a symmetric operator on a maximal domain, we extend the domain from C_0^∞ to include certain functions that are not compactly supported. Given a specific set of twist angles $\{\chi_{\alpha,j}^f\}$, and $\{\Omega_{\alpha,j}^f\}$, we extend the domain of the operator $\partial\!\!\!/_E$ to functions $f = \{f_{\alpha,j}\}$ that are finite linear combinations of single component functions of the form

$$f^{\alpha',j'} = \{f_{\alpha,j}^{\alpha',j'}\} = \delta_{\alpha'\alpha}\delta_{j'j} e^{ikx + iEt} , \tag{V.104}$$

where

$$(k,E) \in \hat{\Sigma}_{\mathcal{T}^f_{\alpha,j}}\ , \quad \text{defined by} \quad \ell k \in 2\pi\mathbb{Z} - \chi^f_{\alpha,j}\ , \qquad \beta E \in \{2\pi\mathbb{Z} - \Omega^f_{\alpha,j}\theta - k\sigma\}\ . \tag{V.105}$$

The functions (V.104) labelled by α', j' are themselves an orthonormal basis for $L^2(\Sigma)$. Furthermore, the components $f^{\alpha',j'}_{\alpha,j}$ of $f^{\alpha',j'}$ satisfy the twist relations

$$f^{\alpha',j'}_{\alpha,j}(x+\ell,t) = e^{-i\chi^f_{\alpha,j}} f^{\alpha',j'}_{\alpha,j}(x,t)\ , \qquad \text{and} \qquad f^{\alpha',j'}_{\alpha,j}(x,t+\beta) = e^{-i\Omega^f_{\alpha,j}\theta} f^{\alpha',j'}_{\alpha,j}(x-\sigma,t)\ . \tag{V.106}$$

Let $\mathcal{D}_{\mathcal{T}_i}$ denote the domain of two-component functions in $\mathfrak{T}_i$ that are finite linear combinations of functions in C_0^∞ with those of the form (V.104)–(V.105). Let

$$\mathcal{D}_{\mathcal{T}} = \oplus_{i=1}^n \mathcal{D}_{\mathcal{T}_i}\ . \tag{V.107}$$

Correspondingly, let $\partial\!\!\!/_{E,\mathcal{T}_i}$ denote the operator $\partial\!\!\!/_{E,i}$ extended to the domain $\mathcal{D}_{\mathcal{T}_i}$, and let $\partial\!\!\!/_{E,\mathcal{T}}$ denote the extension of $\partial\!\!\!/_E$ to the domain $\mathcal{D}_{\mathcal{T}}$. The operator $\partial\!\!\!/_{E,\mathcal{T}_i}$ is not in general symmetric; the condition for symmetry of the full $\partial\!\!\!/_{E,\mathcal{T}}$ is:

Proposition V.8.1. *The operator $\partial\!\!\!/_{E,\mathcal{T}}$ with domain $\mathcal{D}_{\mathcal{T}}$ is symmetric if and only if*

$$e^{i\chi^f_{1,i}} = e^{i\chi^f_{2,i}}\ , \qquad \textit{and} \qquad e^{i\Omega^f_{1,i}\theta} = e^{i\Omega^f_{2,i}\theta}\ , \qquad \textit{for all} \quad 1 \le i \le n\ . \tag{V.108}$$

If (V.108) holds, then $\partial\!\!\!/_{E,\mathcal{T}}$ is essentially self-adjoint.

Proof. Expontial functions of the form (V.104) have the property that the twist relations (V.106) carry over to derivatives. As the representation (V.99) illustrates that $\partial\!\!\!/_{E,\mathcal{T}}$ is off-diagonal in the α-index, the boundary terms that arise from integration by parts of $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial t}$ vanish if and only if the $(1,i)$ and $(2,i)$ components have the same twists. Hence (V.108) is necessary and sufficient for $\partial\!\!\!/_{E,\mathcal{T}}$ to be symmetric on the domain $\mathcal{D}_{\mathcal{T}}$.

In case (V.108) holds, the domain $\mathcal{D}_{\mathcal{T}}$ contains a basis of orthonormal eigenfunctions $g^{j,k,E}_\pm$ for $\partial\!\!\!/_{E,\mathcal{T}}$. The corresponding eigenvalue of $\partial\!\!\!/_{E,\mathcal{T}}$ is $\pm\sqrt{k^2+E^2}$, where $(k,E) \in \hat{\Sigma}_{\mathcal{T}^f_i} = \hat{\Sigma}_{\mathcal{T}^f_{\alpha,i}}$. The eigenvectors have the form

$$\left(g^{i,k,E}_\pm\right)_{\alpha,j}(x,t) = \delta_{ij}\frac{1}{\sqrt{|\Sigma|}} c_{\pm,\alpha,j} e^{ikx+iEt}\ , \tag{V.109}$$

where the coefficents $c_{\pm,\alpha,j}$ are given as follows: let $E \pm ik = re^{\pm i\phi}$, where $r = \sqrt{E^2+k^2} > 0$ denotes the positive square root. Define $(E \pm ik)^{1/2} = r^{1/2}e^{\pm i\phi/2}$, where again $r^{1/2} > 0$, so $e^{\pm i\phi/2} = (E \pm ik)^{1/2}\left(E^2+k^2\right)^{-1/4}$. Choose

$$\begin{pmatrix} c_{+,1,j} \\ c_{+,2,j} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\phi/2} \\ e^{i\phi/2} \end{pmatrix}\ , \qquad \text{and} \qquad \begin{pmatrix} c_{-,1,j} \\ c_{-,2,j} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -e^{-i\phi/2} \\ e^{i\phi/2} \end{pmatrix}\ . \tag{V.110}$$

Since k never vanishes for $(k, E) \in \hat{\Sigma}_{\mathcal{T}_i^f}$, the two choices yield distinct (orthogonal) eigenvectors. By inspection, the vectors (V.109)–(V.110) provide an orthonormal basis of eigenfunctions for each $\not{\partial}_{E,\mathcal{T}_i}$, from which we conclude that the operator $\not{\partial}_{E,\mathcal{T}}$ is essentially self-adjoint. This completes the proof.

Given $\mathcal{T}$, we introduce a twist/translation semi-group $u_{\mathcal{T}}(\theta', x', t')$ that acts on the Hilbert space $\mathfrak{T}$. This group is the natural action of twists and space translations that leave invariant the subpaces of functions $f_{\alpha,i}(x,t)$ with fixed indices α, i. Specifically, we define $u_{\mathcal{T}}$ on functions f that satisfy the twist relations (V.106) for $\mathcal{D}_{\mathcal{T}}$. For $\theta', x' \in \mathbb{R}$ and $t' \in \mathbb{R}_+$, let

$$\left(u_{\mathcal{T}}(\theta', x', t') f\right)_{\alpha,i}(x,t) = \left(u_{\mathcal{T}}(\theta', x', t')_{\alpha,i} f_{\alpha,i}\right)(x,t) = e^{i\Omega_{\alpha,i}\theta'} f_{\alpha,i}(x + x', t - t') \ . \qquad \text{(V.111)}$$

Each vector of the form (V.104) is an simultaneous eigenvector for each of the $u_{\mathcal{T}}(\theta', x', t')$ as θ', x', t' vary. Thus each $u_{\mathcal{T}}(\theta', x', t')$ extends uniquely to all vectors in $\mathfrak{T}$. With this definition, the semi-group $u_{\mathcal{T}}$ leaves the subspace $\mathcal{D}_{\mathcal{T}} \subset \mathfrak{T}$ invariant, where $\mathcal{D}_{\mathcal{T}}$ is defined in (V.106)–(V.107). The parameters $\mathcal{T}$ determine a specific representation of this group, so that when $\theta' = \theta$, $x' = \sigma$, and $t' = \beta$ agrees with the twist and translation of $\mathcal{T}$, or when $\theta' = \chi_{\alpha,i}/\Omega_{\alpha,i}$, $x' = \ell$, and $t' = 0$, we obtain

$$u_{\mathcal{T}}(\theta, \sigma, \beta)_{\alpha,i} = I \qquad \text{and} \qquad u_{\mathcal{T}}(\chi_{\alpha,i}/\Omega_{\alpha,i}, \ell, 0)_{\alpha,i} = I \ , \qquad \text{(V.112)}$$

for all $1 \leq \alpha \leq 2$ and $1 \leq i \leq n$. The special operators (V.112) (and their integer powers) act as the identity on the domain $\mathcal{D}_{\mathcal{T}}$, and they also extend uniquely from this dense domain to act as the identity on all of $\mathfrak{T}$.

Consider the Dirac twist field ψ^{χ} paired with a smooth test function $f \in \mathcal{D}_{\mathcal{T}}$. The field $\psi^{\chi}(f)$ transforms under the action of the two-parameter abelian, group $U^{f,\chi}(\theta', x')$ on $\mathcal{H}^{f,\chi}$, compatibly with the action of the twist semi-group $u_{\mathcal{T}}(\theta', x', 0)$. Namely

$$U^{f,\chi}(\theta', x')\, \psi^{\chi}(f)\, U^{f,\chi}(\theta', x')^* = \psi^{\chi}(u_{\mathcal{T}}(\theta', x', 0) f) \ . \qquad \text{(V.113)}$$

Inspection of the eigenbasis above yields the following:

Proposition V.8.2. *The operator $\not{\partial}_{E,\mathcal{T}}$ commutes with the group $u_{\mathcal{T}}(\theta', x', 0)$ if and only if (V.108) holds.*

Independent of this condition on twists, the operator

$$\not{\partial}_{E,\mathcal{T}_i}{}^* \not{\partial}_{E,\mathcal{T}_i} = -\begin{pmatrix} \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} & 0 \\ 0 & \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \end{pmatrix} \qquad \text{(V.114)}$$

has a diagonal representation on $\mathfrak{T}_i$. Let us denote this operator

$$\Delta_{\mathcal{T}_i} = \not{\partial}_{E,\mathcal{T}_i}{}^* \not{\partial}_{E,\mathcal{T}_i} \ , \qquad \text{and let} \qquad \Delta_{\mathcal{T}} = \oplus_{i=1}^{n} \Delta_{\mathcal{T}_i} = \not{\partial}_{E,\mathcal{T}}{}^* \not{\partial}_{E,\mathcal{T}} \ . \qquad \text{(V.115)}$$

Proposition V.8.3. *Regardless of the restrictions (V.108), the operator $\Delta_{\mathcal{T}}$, with the domain $\mathcal{D}_{\mathcal{T}}$ defined in (V.107) is essentially self-adjoint. Furthermore, the operators $\Delta_{\mathcal{T}}$ and $u_{\mathcal{T}}(\theta', x', 0)$ commute.*

Proof. The functions $g \in \mathcal{D}_{\mathcal{T}_i}$ labelled by $\{\alpha, i, k, E\}$, for $(k, E) \in \hat{\Sigma}_{\mathcal{T}^f_{\alpha,i}}$, and with components equal to

$$g^{\{\alpha,i,k,E\}}_{\alpha',i'}(x,t) = \delta_{ii'}\delta_{\alpha\alpha'}\frac{1}{\sqrt{|\Sigma|}}e^{ikx+iEt} \,, \tag{V.116}$$

are a complete, orthonormal set of eigenfunctions for ${\not\partial_{E,\mathcal{T}_i}}^*\not\partial_{E,\mathcal{T}_i}$. Thus ${\not\partial_{E,\mathcal{T}_i}}^*\not\partial_{E,\mathcal{T}_i}$ (respectively ${\not\partial_{E,\mathcal{T}}}^*\not\partial_{E,\mathcal{T}}$) are essentially self-adjoint on the domains $\mathcal{D}_{\mathcal{T}^f_i} \in \mathfrak{T}_i$ (respectively $\mathcal{D}_{\mathcal{T}} \in \mathfrak{T}$). These eigenfunctions are also eigenfunctions of $U^{f,\chi}(\theta)$, which is a product of commuting operators indexed by α and i. Thus $\Delta_{\mathcal{T}}$ and $U^{f,\chi}(\theta)$ commute, completing the proof.

Let us denote the self-adjoint closures by $\Delta_{\mathcal{T}_i}$ and $\Delta_{\mathcal{T}}$. The eigenvalues of $\Delta_{\mathcal{T}_i}$ are $E^2 + k^2$, with $(k, E) \in \hat{\Sigma}_{\mathcal{T}^f_i}$, and hence as $k \neq 0$, the operator $\Delta_{\mathcal{T}_i}$ has a bounded inverse. Let

$$C_{\mathcal{T}^f_i} = (\Delta_{\mathcal{T}_i})^{-1} \,. \tag{V.117}$$

Here $C_{\mathcal{T}^f_i}$ acts diagonally on $\mathfrak{T} = \oplus_{i=1}^n \mathfrak{T}_i$, namely $C_{\mathcal{T}^f} = \oplus_{i=1}^n C_{\mathcal{T}^f_i}$, with the action on $\mathfrak{T}_i$ given by the 2×2 matrix

$$C^f_{\mathcal{T}_i} = \begin{pmatrix} \Delta^{-1}_{\mathcal{T}_{1,i}} & 0 \\ 0 & \Delta^{-1}_{\mathcal{T}_{2,i}} \end{pmatrix} \,. \tag{V.118}$$

Let us also designate $\mathcal{T}^*$ as $\mathcal{T}$, but with $\mathcal{T}_{1,i}$ interchanged with $\mathcal{T}_{2,i}$, for each $1 \leq i \leq n$. For example,

$$C^f_{\mathcal{T}^*_i} = \begin{pmatrix} \Delta^{-1}_{\mathcal{T}_{2,i}} & 0 \\ 0 & \Delta^{-1}_{\mathcal{T}_{1,i}} \end{pmatrix} \,. \tag{V.119}$$

The second operator we define on $\mathfrak{T}$ is the pair correlation operator $S^\chi_{\mathcal{T}}$. This operator is defined an an integral operator, using as the integral kernels the elements of the pair correlation matrix $S^\chi_{\mathcal{T},\alpha\alpha',ii'}(x - x', t - t')$, defined in (V.80). Define

$$(S^\chi_{\mathcal{T}} f)_{\alpha,i} = \sum_{i'=1}^{n}\sum_{\alpha'=1}^{2}\int_\Sigma S^\chi_{\mathcal{T},\alpha\alpha',ii'}(x - x', t - t') f_{\alpha',i'}(x', t') dx' dt' \,. \tag{V.120}$$

Proposition V.8.4. *The fermionic pair correlation operator is the Green's function for the Euclidean Dirac operator,*

$$S^\chi_{\mathcal{T}} = \left(\not\partial_{E,\mathcal{T}}\right)^{-1} \,. \tag{V.121}$$

Also

$$C^f_{\mathcal{T}} = \left({\not\partial_{E,\mathcal{T}}}^*\not\partial_{E,\mathcal{T}}\right)^{-1} \,, \qquad \not\partial_{E,\mathcal{T}} = \left(\not\partial_{E,\mathcal{T}^*}\right)^* \,. \qquad \text{and} \qquad S^\chi_{\mathcal{T}} = {\not\partial_{E,\mathcal{T}}}^* C^f_{\mathcal{T}^*} \,. \tag{V.122}$$

Proof. We verify the identity

$$\not{\partial}_{E,\mathcal{T}} S^{\chi}_{\mathcal{T}} = I \tag{V.123}$$

by differentiating the representations of Proposition V.7.1. This yields

$$\left(\not{\partial}_E S^{\chi}_{\mathcal{T}}\right)_{\alpha\alpha',ii'}(x-x',t-t') = \delta_{ii'}\delta_{\alpha\alpha'}\delta(x-x')\delta(t-t') \;, \tag{V.124}$$

and thereby (V.121) holds.

VI Fermionic Regularization

VI.1 Massive Fields

We introduce mass $m > 0$ fields for fermions, that correspond to the bosonic massive fields of §III.2.2. We express the fermionic wave functions in terms of the parameter

$$\nu_m(k) = \sqrt{\mu_m(k) + k} \;, \tag{VI.1}$$

where as before $\mu_m(k) = \sqrt{k^2 + m^2}$. Then

$$\nu_m(k)^2 + \nu_m(-k)^2 = 2\mu_m(k) \;, \text{ and } \nu_m(k)\nu_m(-k) = m \;. \tag{VI.2}$$

Also

$$\nu_m(k)^2 - \nu_m(-k)^2 = 2k \tag{VI.3}$$

As in the bosonic case, we define the massive field without a twist. Thus we restrict attention to the momemtum set $K = \{k : k\ell \in 2\pi\mathbb{Z}\} = K^{m=0}_i = K^{f,\chi^f=0}_{\alpha,i} = K^{f,\chi^f=0}_{\pm,i}$. The fields take the form,

$$\psi^m_{1,i}(x) = \frac{1}{\sqrt{\ell}} \sum_{k\in K} \xi^m_{1,i}(k) e^{-ikx} \;, \text{ where } \xi^m_{1,i}(k) = \frac{1}{\sqrt{2\mu_m(k)}} \left(\nu_m(k) b_{+,i}(k)^* + \nu_m(-k) b_{-,i}(-k)\right) \tag{VI.4}$$

and

$$\psi^m_{2,i}(x) = \frac{1}{\sqrt{\ell}} \sum_{k\in K} \xi^m_{2,i}(k) e^{-ikx} \;, \text{ where } \xi^m_{2,i}(k) = \frac{-i}{\sqrt{2\mu_m(k)}} \left(\nu_m(-k) b_{+,i}(k)^* - \nu_m(k) b_{-,i}(-k)\right) \;. \tag{VI.5}$$

From the identities (VI.2), we infer that these fermionic coordinates satisfy the CAR

$$\{\xi^m_{\alpha,i}(k)^{\#}, \xi^m_{\alpha',i'}(k')^{\#'}\} = \delta_{\alpha\alpha'}\delta_{ii'}\delta_{kk'}\delta_{\#^*\#'}\; I \;. \tag{VI.6}$$

As a consequence the fields satisfy the CAR

$$\{\psi^m_{\alpha,i}(x)^{\#}, \psi^m_{\alpha',i'}(x')^{\#'}\} = \delta_{\alpha\alpha'}\delta_{ii'}\delta_{\#^*\#'}\; \delta(x-x')\; I \;. \tag{VI.7}$$

Likewise the identity (VI.3) leads to expressions for the Hamiltonian and momentum operators for the massive fields as integrals of local densities,

$$\begin{aligned} H_0^{f,m} &= \int_0^\ell :\overline{\psi^m}\left(-i\gamma^1\partial_x\right)\psi^m - m:\overline{\psi^m}\,\psi^m:\,dx \\ &= \sum_{i=1}^n\sum_{k\in K}\mu_m(k)\left(b_{+,i}(k)^*b_{+,i}(k)+b_{-,i}(-k)^*b_{-,i}(-k)\right)\ , \end{aligned} \tag{VI.8}$$

and

$$\begin{aligned} P_0^{f,m} &= \int_0^\ell :\overline{\psi^m}\left(-i\gamma^0\partial_x\right)\psi^m:\,dx \\ &= \sum_{i=1}^n\sum_{k\in K}k\left(b_{+,i}(k)^*b_{+,i}(k)-b_{-,i}(-k)^*b_{-,i}(-k)\right)\ . \end{aligned} \tag{VI.9}$$

The massive, real-time free field, with initial data (VI.4)–(VI.5), has the components

$$\psi^m_{\mathrm{RT},\alpha,i}(x,t) = e^{itH_0^{f,m}}\psi^m_{\alpha,i}(x)e^{-itH_0^{f,m}} = e^{itH_0^{f,m}-ixP^{f,m}}\psi^m_{\alpha,i}(0)e^{-itH_0^{f,m}+ixP^{f,m}}\ . \tag{VI.10}$$

This field is the solution to the real-time Dirac equation

$$\left(i\partial\!\!\!/ - m\right)\psi^m_{\mathrm{RT}}(x,t) = 0\ , \tag{VI.11}$$

as can be seen by taking the time derivative of (VI.10).

We may define a global twist generator $J^{f,m}$ for the massive Dirac fields,

$$J^{f,m} = \sum_{i=1}^n\sum_{\alpha=1}^2\Omega^f_{\alpha,i}\int_0^\ell :\psi^m_{\alpha,i}(x)\,\psi^m_{\alpha,i}(x)^*:dx\ . \tag{VI.12}$$

However, unlike in the case of the massless Dirac field, we need to take $\Omega^{f,m}_{1,i} = \Omega^{f,m}_{2,i}$ in order for $J^{f,m}$ to be a symmetry of $H_0^{f,m}$. In particular, this requirement ensures that the symmetry will leave the mass term $m\int :\overline{\psi^m_i}\,\psi^m_i:\,dx$ in (VI.8) invariant.

VI.2 Dirac String Fields

The Dirac string fields $\psi^{\mathrm{str}}(x,t)$ are the zero-mass limits of the massive Dirac fields $\psi^m(x,t)$ of §VI.1. Remark that

$$\lim_{m\to 0}\frac{\nu_m(k)}{\sqrt{2\mu_m(k)}} = \begin{cases} 1\ , & \text{if } k>0 \\ \frac{1}{\sqrt{2}}\ , & \text{if } k=0 \\ 0\ , & \text{if } k<0 \end{cases}\ , \tag{VI.13}$$

and the limits $m \to 0$ and $k \to 0$ in (VI.13) cannot be interchanged. Thus

$$\psi_{1,i}^{\rm str}(x) = \frac{1}{\sqrt{\ell}} \sum_{k\in K} \xi_{1,i}^{\rm str}(k) e^{-ikx} = \frac{1}{\sqrt{\ell}} \xi_{1,i}^{\rm str} + \frac{1}{\sqrt{\ell}} \sum_{\substack{k>0 \\ k\in K}} b_{+,i}(k)^* e^{-ikx} + \frac{1}{\sqrt{\ell}} \sum_{\substack{k<0 \\ k\in K}} b_{-,i}(-k) e^{-ikx} \;, \qquad \text{(VI.14)}$$

and

$$\psi_{2,i}^{\rm str}(x) = \frac{1}{\sqrt{\ell}} \sum_{k\in K} \xi_{2,i}^{\rm str}(k) e^{-ikx} = \frac{1}{\sqrt{\ell}} \xi_{2,i}^{\rm str} + \frac{1}{\sqrt{\ell}} \sum_{\substack{k<0 \\ k\in K}} b_{+,i}(k)^* e^{-ikx} - \frac{1}{\sqrt{\ell}} \sum_{\substack{k>0 \\ k\in K}} b_{-,i}(-k) e^{-ikx} \;, \qquad \text{(VI.15)}$$

so

$$\xi_{1,i}^{\rm str}(k) = \begin{cases} b_{+,i}(k)^* \;, & \text{if } k>0 \\ \frac{1}{\sqrt{2}} \left(b_{+,i}(0)^* + b_{-,i}(0) \right) \;, & \text{if } k=0 \\ b_{-,i}(-k) \;, & \text{if } k<0 \end{cases} \;, \qquad \text{(VI.16)}$$

and

$$\xi_{2,i}^{\rm str}(k) = \begin{cases} -b_{-,i}(-k) \;, & \text{if } k>0 \\ \frac{1}{\sqrt{2}} \left(b_{+,i}(0)^* - b_{-,i}(0) \right) \;, & \text{if } k=0 \\ b_{+,i}(k)^* \;, & \text{if } k<0 \end{cases} \;. \qquad \text{(VI.17)}$$

As in the other cases, the CAR for the string coordinates are

$$\{ \xi_{\alpha,i}^{\rm str}(k)^{\#}, \xi_{\alpha',i'}^{\rm str}(k')^{\#'} \} = \delta_{\alpha\alpha'} \delta_{ii'} \delta_{kk'} \delta_{\#^*\#'} \; I \;, \qquad \text{(VI.18)}$$

giving

$$\{ \psi_{\alpha,i}^{\rm str}(x)^{\#}, \psi_{\alpha',i'}^{\rm str}(x')^{\#'} \} = \delta_{\alpha\alpha'} \delta_{ii'} \delta_{\#^*\#'} \; \delta(x-x') \; I \;. \qquad \text{(VI.19)}$$

VII $N=2$ Supersymmetry

Consider the initial $t=0$ data for an n-component complex scalar field $\varphi^\chi = \{\varphi_j^\chi : 1 \le j \le n\}$, and n-copies of a 2-component Dirac field $\psi^\chi = \{\psi_{\alpha,j}^\chi : 1 \le \alpha \le 2, \text{ and } 1 \le j \le n\}$. We define these fields on the cover $\mathbb{R}$ of the circle S^1 with period ℓ. We assume that they satisfy that satisfy the twist relations

$$\varphi_j^\chi(x+\ell) = e^{i\chi_j^b}\, \varphi_j^\chi(x) \;, \qquad \text{and} \qquad \psi_{\alpha,j}^\chi(x+\ell) = e^{i\chi_{\alpha,j}^f}\, \psi_{\alpha,j}^\chi(x) \;. \qquad \text{(VII.1)}$$

Such fields are *twisted periodic*, with period ℓ, and with twisting angles $\{\chi\} = \{\chi_j^b, \chi_{\alpha,j}^f\}$.

Take these fields together, acting on the tensor product Hilbert space $\mathcal{H} = \mathcal{H}^b \otimes \mathcal{H}^f$. Denote the lattice of bosonic momenta and the lattice of fermionic momenta for the components and the copies of the bosonic and fermionic fields by

$$K^b = \{K_1^b, \ldots, K_n^b\} \;, \qquad K_1^f = \{K_{1,1}^f, \ldots, K_{1,n}^f\} \;, \qquad \text{and} \qquad K_2^f = \{K_{2,1}^f, \ldots, K_{2,n}^f\} \;. \quad \text{(VII.2)}$$

This replaces the notation K_j^{χ}, $K_{\alpha,j}^{\chi^f}$, $K_{\pm,j}^{\chi^f}$, etc. used in §II and §V.

Let $P^{b,\chi}$ and $P^{f,\chi}$ denote the bosonic and fermionic momentum operators defined in (II.19) and (V.41), and denote the total momentum operator as

$$P = P^{b,\chi} \otimes I + I \otimes P^{f,\chi} = P^{b,\chi} + P^{f,\chi} . \tag{VII.3}$$

In order to simplify notation, if the operator P^b is defined on $\mathcal{H}^b$, then denote the operator $P^b \otimes I$ acting on $\mathcal{H}$ also by P^b, and likewise for other operators on $\mathcal{H}^b$ or $\mathcal{H}^f$. The operator P generates a unitary translation group $e^{ix'P}$ that acts on the fields by $e^{ix'P}\varphi^\chi(x)e^{-ix'P} = \varphi^\chi(x-x')$ and also $e^{ix'P}\psi^\chi(x)e^{-ix'P} = \psi^\chi(x-x')$.

We study densities $D(x)$ that are functions of $\varphi^\chi(x)$ and of $\psi^\chi(x)$. We say that $D(x)$ is translation covariant, if

$$e^{ix'P}D(x)e^{-ix'P} = D(x-x') . \tag{VII.4}$$

We also assume that the densities we study obey a spatial twist relation of the form

$$D(x+\ell) = e^{i\vartheta}D(x) , \tag{VII.5}$$

where ϑ is a real constant depending on the specific density $D(x)$. In other words, $D(x)$ is twisted periodic with period ℓ and twisting angle ϑ.

We wish to integrate the density $D(x)$ over a period of length ℓ to obtain a charge D. In order to get a well-behaved charge, we modify the density $D(x)$ by forcing it to be periodic. Namely we take the charge density to be $D(x)e^{-ix\vartheta/\ell}$, and define the charge D by

$$D = \int_0^\ell D(x)e^{-i\vartheta x/\ell}\,dx . \tag{VII.6}$$

The charges D that we study generally have the property of a cohomology operator,

$$D^2 = 0 . \tag{VII.7}$$

Since D is the integral of a periodic density, integrating the density over any interval $[a, a+\ell]$ of length ℓ would yield the same D. However, shifting the interval does not correspond to the action of the unitary translation group e^{-iaP} on $\mathcal{H}$. In fact D is invariant under spatial translations generated by P, only if $\vartheta = 0$. The translation group e^{-iaP} acts on D as

$$e^{-iaP}De^{iaP} = e^{ia\vartheta/\ell}D , \tag{VII.8}$$

as follows from expanding the left side of (VII.5) as a power series in a, and summing this series using

$$[-iP, D] = \int_0^\ell \frac{\partial D(x)}{\partial x}e^{-ix\vartheta/\ell}\,dx = i\frac{\vartheta}{\ell}D . \tag{VII.9}$$

The second equality in (VII.9) results from integration by parts.

Let $J = J^{b,\chi} + J^{f,\chi}$ be the total twist generator; this is defined as the sum of the bosonic and fermionic generators of twists introduced in (II.20) and (V.52). We also will assume that our densities $D(x)$ transform under twists as

$$e^{i\theta J} D(x) e^{-i\theta J} = e^{i\theta\Lambda} D(x) \; , \tag{VII.10}$$

where Λ is another constant depending on $D(x)$. Then the two-parameter unitary group $U(\theta, \sigma)$ of twists defined by

$$U(\theta, \sigma) = e^{iJ\theta + i\sigma P} \; , \tag{VII.11}$$

acts on D as

$$U(\theta, \sigma) D U(\theta, \sigma)^* = e^{i\theta\Lambda - i\sigma\vartheta/\ell} D \; . \tag{VII.12}$$

The charge D is invariant under the full group $U(\theta, \sigma)$ if and only if $\Lambda = \vartheta = 0$. Equivalently, D is invariant under the action of the group $U(\theta, \sigma)$, if for all x, θ,

$$e^{i\theta J} D(x + \ell) e^{-i\theta J} = D(x) \; . \tag{VII.13}$$

Such densities are *twist invariant and ℓ-periodic.*

We are mainly interested in charges Q that are symmetric (and essentially self adjoint). We obtain symmetric charges as the real or imaginary parts of D. In particular, define the charge $Q = D + D^*$, and the second charge $\widetilde{Q} = -i\,(D - D^*)$, where where D is a charge of the form above. If we assume $D^2 = 0$, as remarked in (VII.7), then also $(D^*)^2 = 0$ and

$$Q^2 = \frac{1}{2}\{Q, Q\} = D^* D + D D^* = \frac{1}{2}\{\widetilde{Q}, \widetilde{Q}\} = \widetilde{Q}^2 \; . \tag{VII.14}$$

Furthermore Q and $\widetilde{Q}$ are automatically independent, in the sense that

$$\{Q, \widetilde{Q}\} = -i\{D + D^*, D - D^*\} = 0 \; . \tag{VII.15}$$

In the following subsection, we introduce a Hamiltonian H for a class of $N = 2$ supersymmetric interactions. These examples have two densities $D_1(x)$ and $D_2(x)$ of the above type, yielding charges D_1 and D_2. The charge D_1 yields two indepedent, symmetric supercharges Q_1 and $\widetilde{Q}_1$, defined as

$$Q_1 = D_1 + D_1^* \; , \qquad \text{and} \qquad \widetilde{Q}_1 = -i\,(D_1 - D_1^*) \; . \tag{VII.16}$$

Furthermore, the charges Q_1 and $\widetilde{Q}_1$ are square roots of $H + P$,

$$Q_1^2 = \widetilde{Q}_1^2 = H + P \; , \qquad \text{and} \qquad \{Q_1, \widetilde{Q}_1\} = 0 \; . \tag{VII.17}$$

The first identity also can be written

$$H + P = D_1^* D_1 + D_1 D_1^* \; . \tag{VII.18}$$

The charge D_1 occurs as the integral of a twist-invariant, ℓ-periodic density $D_1(x)$. Thus D_1 will be invariant under twists and under translations. As a consequence,

$$U(\theta,\sigma)D_1 = D_1U(\theta,\sigma) \;, \qquad \text{and} \quad \mathrm{U} \;\, (\theta,\sigma)D_1^* = D_1^*U(\theta,\sigma) \;. \tag{VII.19}$$

Thus the charge $H+P$ commutes with $U(\theta,\sigma)$. Assuming P also commutes with $U(\theta,\sigma)$, it follows that H commutes with $U(\theta,\sigma)$,

$$U(\theta,\sigma)H = HU(\theta,\sigma) \;. \tag{VII.20}$$

The charge D_2 yields two symmetric supercharges $Q_2 = D_2 + D_2^*$ and $\widetilde{Q}_2 = -i\,(D_2 - D_2^*)$. The charge D_2 has the property $D_2^2 = 0$, so

$$Q_2^2 = \widetilde{Q}_2^2 \;, \qquad \text{and} \qquad \{Q_2, \widetilde{Q}_2\} = 0 \;. \tag{VII.21}$$

These charges are related to H and P by

$$Q_2^2 = \widetilde{Q}_2^2 = H - P + \phi\mathcal{R} \;. \tag{VII.22}$$

Here $\mathcal{R}$ is an error term, not in the usual supersymmetry algebra. (Of course we could have chosen different twists so that $Q_2^2 = H - P$, with the error term appearing in the expression for Q_1^2.) The error term $\mathcal{R}$ is translation and twist invariant,

$$U(\theta,\sigma)\mathcal{R} = \mathcal{R}U(\theta,\sigma) \;. \tag{VII.23}$$

The operator $\mathcal{R}$ turns out to be a difference of two fermionic number operators, see (VII.54). It is independent of W, and it depends on the twist angles χ only implicitly through the choice of the Fourier momenta. Furthermore it satisfies an *a priori* estimate of the form

$$\pm\mathcal{R} \le M(H+1) \;, \tag{VII.24}$$

where M is a constant. As a consequence, the domain of $H^{1/2}$ provides a form domain for $\mathcal{R}$. Hence using (VII.17) we infer that

$$-P \le H \;, \qquad \text{and} \qquad P \le H + \phi\,(M+1)\,(H+I) \;\; . \tag{VII.25}$$

Thus we can use the representations

$$H = \frac{1}{2}\left(Q_1^2 + Q_2^2\right) - \frac{1}{2}\phi\mathcal{R} = \frac{1}{2}\left(\widetilde{Q}_1^2 + \widetilde{Q}_2^2\right) - \frac{1}{2}\phi\mathcal{R} \;, \tag{VII.26}$$

and

$$P = \frac{1}{2}\left(Q_1^2 - Q_2^2\right) + \frac{1}{2}\phi\mathcal{R} \;. \tag{VII.27}$$

This error arises because the density $Q_2(x)$ has the form $Q_2(x) = D_2(x)e^{-ix\phi/\ell} + D_2(x)^*e^{ix\phi/\ell}$, where

$$e^{i\theta J}e^{-i\ell P}D_2(x)e^{i\ell P}e^{-i\theta J} = e^{i\theta}\,D_2(x+\ell) = e^{i\theta+i\phi}\,D_2(x) \;. \tag{VII.28}$$

Unlike the first pair of charges, the pair of charges Q_2 and $\widetilde{Q}_2$ are *neither* translation *nor* twist-invariant under the action of $U(\theta,\sigma)$. As a consequence of (VII.28) and (VII.12),

$$U(\theta,\sigma)D_2U(\theta,\sigma)^* = e^{i\theta - i\phi\sigma/\ell} D_2\ . \tag{VII.29}$$

The different components of each pair of charges are independent, in the sense that

$$\{Q_1, \widetilde{Q}_1\} = 0 = \{Q_2, \widetilde{Q}_2\}\ . \tag{VII.30}$$

In the spirit of $N = 2$ supersymmetry, we would also like the pair of charges Q_1, $\widetilde{Q_1}$ to be independent of the second pair Q_2, $\widetilde{Q_2}$. However, we find that

$$\{Q_1, Q_2\} = \{\widetilde{Q}_1, \widetilde{Q}_2\} = \phi\left(\widetilde{\mathcal{R}} + \widetilde{\mathcal{R}}^*\right)\ , \tag{VII.31}$$

and

$$\{Q_1, \widetilde{Q}_2\} = \{\widetilde{Q}_1, Q_2\} = -i\phi\left(\widetilde{\mathcal{R}} - \widetilde{\mathcal{R}}^*\right)\ . \tag{VII.32}$$

Here $\widetilde{\mathcal{R}}$ is a second error term, and in our examples, the operator $\widetilde{\mathcal{R}}$ is given in (VII.55). The error term $\widetilde{\mathcal{R}}$, like the error term $\mathcal{R}$, it is amenable to estimates. Thus we may also use the representation

$$H = \frac{1}{2}\left(Q_1 + Q_2\right)^2 - \frac{1}{2}\phi\left(\mathcal{R} + \widetilde{\mathcal{R}} + \widetilde{\mathcal{R}}^*\right)\ , \tag{VII.33}$$

claimed in (I.18). In this relation H is invariant under the twist-translation group $U(\theta,\sigma)$, but neither Q_2 nor $\widetilde{\mathcal{R}} + \widetilde{\mathcal{R}}^*$ commutes with $U(\theta,\sigma)$.

VII.1 Supercharges

The charges Q_α and $\widetilde{Q}_\alpha$ exist both for free (non-interacting) fields, as well as for certain non-linear supersymmetric interactions between bosons and fermions (generalized Yukawa interactions). Wess and Zumino introduced such models (without twists); they are parameterized by a polynomial $W(z)$ called the *superpotential*. In the physics literature, the interactions we study are also called "Landau-Ginsburg" interactions. We analyze the properties of the supercharges Q both for non-interacting fields and for interactions with an ultraviolet regularization.

We make three basic assumptions, two on the superpotential and one on the twisting angles. These assumptions are identical to, or elaborations of, the assumptions in §III. We require that the superpotential $W(z)$ satisfy the following:

(QH) The function $W(z)$ is a holomorphic, quasi-homogeneous polynomial with weights Ω as defined in (III.5) and (III.7). The weights must lie in the interval $\Omega_i \in (0, \frac{1}{2}]$.

(EL) The function $W(z)$ satisfies the elliptic stability bounds (III.17) and (III.18).

In order to obtain densities with the desired twist relations, we begin by making some global restrictions concerning the bosonic and the fermionic twisting angles $\{\chi\} = \{\chi^b, \chi^f\}$, relating these angles to the weights Ω and to eachother. Let $\phi \in \mathbb{R}$ denote a real parameter. We state the twist assumption (TA) on the twisting angles:

(TA) The bosonic and fermionic twisting angles are all proportional to one real parameter ϕ, and satisfy the relations

$$\chi^b_j = \Omega_j \phi \ , \qquad \chi^f_{1,j} = \Omega_j \phi \ , \qquad \text{and} \qquad \chi^f_{2,j} = (1 - \Omega_j)\phi \ . \tag{VII.34}$$

In addition, the angles involved in the symmetry generator $J = J^{b,\chi} + J^{f,\chi}$ of twists is specified by the bosonic generator (II.20) and the fermionic generator (V.52). The bosonic and fermionic weights $\{\Omega\} = \{\Omega^b_i \ , \ \Omega^f_{1,i} \ , \ \Omega^f_{2,i}\}$ are chosen as

$$\Omega^b_i = \Omega_i \ , \qquad \Omega^f_{1,i} = \Omega_i \ , \qquad \text{and} \qquad \Omega^f_{2,i} = 1 - \Omega_i \ . \tag{VII.35}$$

An immediate consequence of (TA) is the fact that

$$\chi^f_{1,j} + \chi^f_{2,j} = \phi \ , \tag{VII.36}$$

is j-independent. This restricts the allowed bosonic and fermionic sets of momenta (II.6) and (V.20), so in particular

$$K^b_j = K^f_{1,j} \ , \ \text{ for } \ 1 \le j \le n \ . \tag{VII.37}$$

Furthermore, if $k_1 \in K^f_{1,j} = K^b_j$ and $k_2 \in K^f_{2,j}$, then

$$(k_1 + k_2)\ell = 2\pi(n_1 + n_2) - \phi \ , \ \text{ where } \ n_1,\, n_2 \in \mathbb{Z} \ , \qquad \text{so} \qquad e^{-i(k_1+k_2)x} e^{-ix\phi/\ell} = e^{-2\pi i(n_1+n_2)x/\ell} \ , \tag{VII.38}$$

with ϕ the parameter in (VII.34).

Define the densities

$$D_1(x) = i \sum_{j=1}^{n} \psi^{\chi}_{1,j}(x) \left(\pi^{\chi}_j(x) - \partial_x \overline{\varphi}^{\chi}_j(x)\right) + \sum_{j=1}^{n} \psi^{\chi}_{2,j}(x) \overline{W_j(\varphi^{\chi}(x))} \tag{VII.39}$$

and

$$D_2(x) = i \sum_{j=1}^{n} \psi^{\chi}_{2,j}(x) \left(\overline{\pi}^{\chi}_j(x) + \partial_x \varphi^{\chi}_j(x)\right) + \sum_{j=1}^{n} \psi^{\chi}_{1,j}(x) W_j(\varphi^{\chi}(x)) \ . \tag{VII.40}$$

The following properties follow immediately.

Proposition VII.1.1. *Assume that the potential W satisfies (QH) and that the twist angles obey the restrictions (TA) of (VII.34). Then*

i. *The densities (VII.39)–(VII.40) satisfy the twist relations*

$$e^{i\theta J}D_1(x+\ell)e^{-i\theta J} = D_1(x)\ , \qquad \textit{and} \qquad e^{i\theta J}D_2(x+\ell)e^{-i\theta J} = e^{i\theta+i\phi}D_2(x)\ , \tag{VII.41}$$

leading to the charges

$$D_1 = \int_a^{a+\ell} D_1(x)dx\ , \qquad \textit{and} \qquad D_2 = \int_a^{a+\ell} D_2(x)e^{-ix\phi/\ell}dx\ , \tag{VII.42}$$

that are independent of $a \in \mathbb{R}$. *(We take* $a = 0$*).*

ii. *The charges* D_1 *and* D_2 *transform under* $U(\theta,\sigma)$ *as follows,*

$$U(\theta,\sigma)D_1U(\theta,\sigma)^* = D_1\ , \qquad \textit{and} \qquad U(\theta,\sigma)D_2U(\theta,\sigma)^* = e^{i\theta-i\sigma\phi/\ell}D_2\ . \tag{VII.43}$$

Proposition VII.1.2. *Assume the* W *satisfies (QH) and that the twist angles obey the restrictions (TA) of (VII.34). Then*

$$\begin{aligned} D_1 \;=\; & \sum_{i=1}^n \sum_{k\in K_i^b} \Big(b_{+,i}^{\chi^f}(k)^* a_{+,i}^{\chi}(k)\nu(k) - b_{-,i}^{\chi^f}(-k) a_{-,i}^{\chi}(-k)^*\nu(-k)\Big) \\ & + \sum_{i=1}^n \int_0^\ell \psi_{2,i}^{\chi}\overline{W_i(\varphi^\chi(x))}\, dx\ , \end{aligned} \tag{VII.44}$$

and

$$\begin{aligned} D_2 \;=\; & -i\sum_{i=1}^n \sum_{\substack{k'\in K_i^b \\ k=-(k'+\phi/\ell)}} \Big(b_{+,i}^{\chi^f}(k)^* a_{-,i}^{\chi}(-k')\nu(k') + b_{-,i}^{\chi^f}(-k) a_{+,i}^{\chi}(k')^*\nu(-k')\Big) \\ & + \sum_{i=1}^n \int_0^\ell \psi_{1,i}^{\chi} W_i(\varphi^\chi(x)) e^{-ix\phi/\ell}\, dx\ . \end{aligned} \tag{VII.45}$$

Proof. The representations for D_1 and D_2 are a consequence of the Fourier representations (II.11), (II.13), (V.30), and (V.31), combined with the relations (VII.38), (VII.39), and (VII.40).

The charges D_j are densely defined sesqui-linear forms, with the domain $\mathcal{D}_0$; see for example [5]. Thus we may also define the symmetric charges Q_1 and $\widetilde{Q_1}$ as (twice) the real and imaginary parts of D_1, and likewise Q_2 and $\widetilde{Q_2}$ as (twice) the real and imaginary parts of and D_2, namely

$$Q_1 = D_1 + D_1^*\ , \qquad \widetilde{Q}_1 = -i\left(D_1 - D_1^*\right)\ , \tag{VII.46}$$

$$Q_2 = D_2 + D_2^*\ , \qquad \widetilde{Q}_2 = -i\left(D_2 - D_2^*\right)\ . \tag{VII.47}$$

The free charges also define operators, however the domain questions are straightforward only in the free case with $W = 0$. In order to investigate the charges nonzero W, we need to regularize these expressions. We require regularized Dirac fields, in analogy with the regularized bosonic fields introduced in (IV.47). Define the mollifiers $\mathcal{K}^f_{\Lambda,\alpha,j}$ implicitly by the relations

$$\psi^\chi_{\Lambda,1,j}(x) = \int_0^\ell \mathcal{K}^f_{\Lambda,1,j}(x-y)\psi^\chi_{1,j}(y)dy = \frac{1}{\sqrt{\ell}}\sum_{k\in\mathcal{K}^f_{1,j}} \xi^\chi_{1,j}(k)\hat{\mathcal{K}}(k/\Lambda)e^{-ikx}\ , \tag{VII.48}$$

$$\begin{aligned}\psi^\chi_{\Lambda,2,j}(x) &= \int_0^\ell \mathcal{K}^f_{\Lambda,2,j}(x-y)\psi^\chi_{2,j}(y)dy \\ &= \frac{1}{\sqrt{\ell}}\sum_{k\in\mathcal{K}^f_{2,j}} \xi^\chi_{2,j}(k)\hat{\mathcal{K}}\left((k/\Lambda+(1-2\Omega_j)\,\phi/\ell\Lambda)\right)e^{-ikx}\ .\end{aligned} \tag{VII.49}$$

Here the bosonic momenta lie in $K^\chi_j = \{k : k\ell_j \in 2\pi\mathbb{Z} - \Omega_j\phi\}$, while the fermionic momenta $K^f_{\alpha,j}$ satisfy $K^f_{1,j} = K^\chi_j = K^f_{2,j} + (1-2\Omega_j)\phi$. Thus

$$\begin{aligned}\overline{\mathcal{K}_{\Lambda,j}(x)} &= \mathcal{K}_{\Lambda,j}(-x) = e^{i2\Omega_j\phi x/\ell}\,\mathcal{K}_{\Lambda,j}(x)\ ,\\ \mathcal{K}^f_{\Lambda,1,j}(x) &= \mathcal{K}_{\Lambda,j}(x)\ , \qquad \text{and}\\ \mathcal{K}^f_{\Lambda,2,j}(x) &= e^{i(1-2\Omega_j)\phi x/\ell}\,\mathcal{K}_{\Lambda,j}(x)\ ,\end{aligned} \tag{VII.50}$$

where $\mathcal{K}_{\Lambda,j}$ is the bosonic mollifier (IV.50). With this definition, the components of the fields only depend on the values of $\hat{\mathcal{K}}$ at the bosonic momenta K^χ_j.

We also introduce regularized supercharge densities with the regularization in the interaction terms,

$$D_{\Lambda,1}(x) = i\sum_{j=1}^n \psi^\chi_{1,j}(x)\left(\pi^\chi_j(x) - \partial_x\overline{\varphi}^\chi_j(x)\right) + \sum_{j=1}^n \psi^\chi_{\Lambda,2,j}(x)\overline{W_j(\varphi^\chi_\Lambda(x))}\ , \tag{VII.51}$$

$$D_{\Lambda,2}(x) = i\sum_{j=1}^n \psi^\chi_{2,j}(x)\left(\overline{\pi}^\chi_j(x) + \partial_x\varphi^\chi_j(x)\right) + \sum_{j=1}^n \psi^\chi_{\Lambda,1,j}(x)W_j(\varphi^\chi_\Lambda(x))\ . \tag{VII.52}$$

Define the regularized Hamiltonian for the generalized Yukawa interaction determined by the quasi-homogeneous, holomorphic polynomial W as

$$\begin{aligned}H_\Lambda &= H_\Lambda(W)\\ &= H_0^{b,\chi} + H_0^{f,\chi} + \sum_{j=1}^n\int_0^\ell\int_0^\ell \overline{W_j(\varphi_{\Lambda,j}(x))}\ v_{\Lambda,j}(x-y)\ W_j(\varphi_{\Lambda,j}(y))\,dx\,dy\\ &\quad + \sum_{i,j=1}^n\int_0^\ell \psi^\chi_{\Lambda,1,i}(x)\psi^\chi_{\Lambda,2,j}(x)^*W_{ij}(\varphi^\chi_\Lambda(x))\,dx + \sum_{i,j=1}^n\int_0^\ell \psi^\chi_{\Lambda,2,i}(x)\psi^\chi_{\Lambda,1,j}(x)^*\overline{W_{ij}(\varphi^\chi_\Lambda(x))}\,dx\ ,\end{aligned} \tag{VII.53}$$

where $v_{\Lambda,j}(x)$ is given by (IV.54). Also take the momentum operator P to be given by (VII.3). In addition, define

$$\mathcal{R} = -\frac{2}{\ell}\sum_{i=1}^{n}\int_0^\ell :\psi_{2,i}^{\chi}(x)\psi_{2,i}^{\chi}(x)^*: dx = \frac{2}{\ell}\sum_{i=1}^{n}\left(\sum_{\substack{k\in K_{2,i}^f\\ k>0}} b_{-,i}^{\chi}(-k)^* b_{-,i}^{\chi}(-k) - \sum_{\substack{k\in K_{2,i}^f\\ k<0}} b_{+,i}^{\chi}(k)^* b_{+,i}^{\chi}(k)\right) , \tag{VII.54}$$

and

$$\widetilde{\mathcal{R}} = -\frac{2}{\ell}\int_0^\ell W(\varphi_\Lambda(x)) e^{-ix\phi/\ell}\, dx \,. \tag{VII.55}$$

Proposition VII.1.3. *Assume W satisfies (QH) and that the twist angles obey the restrictions (TA) of (VII.34).*

i. *Then the charges $D_{\Lambda,1}$ and $D_{\Lambda,2}$ are nilpotents,*

$$D_{\Lambda,1}^2 = D_{\Lambda,2}^2 = 0 \,. \tag{VII.56}$$

ii. *The charge $D_{\Lambda,1}$ yields $H_\Lambda + P$, and the charge $D_{\Lambda,2}$ approximately yields $H_\Lambda - P$ through the relations*

$$\{D_{\Lambda,1}, D_{\Lambda,1}^*\} = H_\Lambda + P \,, \qquad \textit{and} \qquad \{D_{\Lambda,2}, D_{\Lambda,2}^*\} = H_\Lambda - P + \phi\mathcal{R} \,, \tag{VII.57}$$

where $\mathcal{R}$ is given in (VII.54). In addition,

$$U(\theta,\sigma)\mathcal{R} = \mathcal{R}U(\theta,\sigma) \,. \tag{VII.58}$$

iii. *The charges $D_{\Lambda,1}$ and $D_{\Lambda,2}$ are approximately independent in the sense that*

$$\{D_{\Lambda,1}, D_{\Lambda,2}\} = 0 \,, \qquad \textit{and} \qquad \{D_{\Lambda,1}^*, D_{\Lambda,2}\} = \phi\widetilde{\mathcal{R}} \,, \tag{VII.59}$$

where $\widetilde{\mathcal{R}}$ is defined in (VII.55).

Proof. Without the ultra-violet mollifiers like $\mathcal{K}_{\Lambda\,,\,j}$, the supercharge forms D_j have no obvious operator domains. The important fact is that when we use the mollified fields defined above, we obtain operator domains for $D_{\Lambda\,,\,j}$, on which the anti-commutators determine sesqui-linear forms. We need to verify that the mollifiers combine in a way that leads to the anti-commutation relations stated in the proposition. We refrain from giving complete details, but in order to illustrate the computations involved we give two sample calculations. For the first illustration we show that $\{D_{\Lambda\,,\,1}\,,\, D_{\Lambda\,,\,1}^*\} = H_\Lambda + P$, as claimed in (VII.57). The commutation relations that involve only the free parts of the $D_{\Lambda\,,\,j}$'s does not involve the mollifiers in question, so we only check terms that involve the potential function W. Therefore, we calculate $X = \{D_{\Lambda\,,\,1}\,,\, D_{\Lambda\,,\,1}^*\} - H_0 - P$, namely

$$X = \sum_{1\le j,j'\le n}\int_0^\ell\int_0^\ell F_{jj'}(x,y)\, dxdy \,, \tag{VII.60}$$

where

$$
\begin{aligned}
F_{jj'}(x,y) &= \Big\{\psi^{\chi}_{\Lambda,2,j}(x)\,\overline{W_j(\varphi^{\chi}_{\Lambda}(x))}\,,\,\psi^{\chi}_{\Lambda,2,j'}(y)^*\,W_{j'}(\varphi^{\chi}_{\Lambda}(y))\Big\} \\
&\quad + \Big\{i\psi^{\chi}_{1,j}(x)\,\pi^{\chi}_{j}(x)\,,\,\psi^{\chi}_{\Lambda,2,j'}(y)^*\,W_{j'}(\varphi^{\chi}_{\Lambda}(y))\Big\} \\
&\quad + \Big\{\psi^{\chi}_{\Lambda,2,j}(x)\,\overline{W_j(\varphi^{\chi}_{\Lambda}(x))}\,,\,i\psi^{\chi}_{1,j'}(y)^*\,\overline{\pi}^{\chi}_{j'}(y)\Big\}\ . \qquad \text{(VII.61)}
\end{aligned}
$$

We claim that the first anti-commutator in (VII.61) equals the bosonic self-interaction term in (VII.53). Using the canonical anti-commutation relations (V.36) and the definition of the fermionic mollifier (VII.49), compute

$$
\Big\{\psi^{\chi}_{\Lambda,2,j}(x)\,\overline{W_j(\varphi^{\chi}_{\Lambda}(x))}\,,\,\psi^{\chi}_{\Lambda,2,j'}(y)^*\,W_{j'}(\varphi^{\chi}_{\Lambda}(y))\Big\} = \overline{W_j(\varphi^{\chi}_{\Lambda}(x))}\,\delta^{f}_{\Lambda,j,j'}(x,y)\,W_{j'}(\varphi^{\chi}_{\Lambda}(y))\ , \qquad \text{(VII.62)}
$$

where

$$
\delta^{f}_{\Lambda,j,j'}(x,y) = \Big\{\psi^{\chi}_{\Lambda,2,j}(x)\,,\,\psi^{\chi}_{\Lambda,2,j'}(y)^*\Big\} = \delta_{jj'}\int_0^{\ell}\mathcal{K}^{f}_{\Lambda,2,j}(x-u)\,\overline{\mathcal{K}^{f}_{\Lambda,2,j}(y-u)}\,du\ . \qquad \text{(VII.63)}
$$

Taking into account the relation (VII.51) and the definition (IV.50), we obtain

$$
\begin{aligned}
\delta^{f}_{\Lambda,j,j'}(x,y) &= \delta_{jj'}\,e^{i(1-2\Omega_j)(x-y)\phi/\ell}\int_0^{\ell}\mathcal{K}_{\Lambda,j}(x-u)\,\overline{\mathcal{K}_{\Lambda,j}(y-u)}\,du \\
&= \delta_{jj'}\,e^{i(1-2\Omega_j)(x-y)\phi/\ell}\left(\frac{1}{\ell}\sum_{k\in K^{\chi}_{j}}|\hat{\mathcal{C}}(k/\Lambda)|^2\,e^{-ik(x-y)}\right)\ , \qquad \text{(VII.64)}
\end{aligned}
$$

showing that $\delta^{f}_{\Lambda,j,j}(x,y)$ equals the kernel $v_{\Lambda,j}(x-y)$ defined in (IV.54). Integrating the expression (VII.62), and summing over j, j', we obtain

$$
\begin{aligned}
&\sum_{1\le j,j'\le n}\int_0^{\ell}dx\int_0^{\ell}dy\,\Big\{\psi^{\chi}_{\Lambda,2,j}(x)\,\overline{W_j(\varphi^{\chi}_{\Lambda}(x))}\,,\,\psi^{\chi}_{\Lambda,2,j'}(y)^*\,W_{j'}(\varphi^{\chi}_{\Lambda}(y))\Big\} \\
&\qquad = \sum_{j=1}^{n}\int_0^{\ell}dx\int_0^{\ell}dy\,\overline{W_j(\varphi^{\chi}_{\Lambda}(x))}\,v_{\Lambda,j}(x-y)\,W_j(\varphi^{\chi}_{\Lambda}(y))\ , \qquad \text{(VII.65)}
\end{aligned}
$$

which is the bosonic self-interaction in (VII.53), as claimed.

The other two anti-commutators in (VII.61) give rise to the boson-fermion interaction terms in (VII.53). In the case of the second anti-commutator, the density is

$$
\begin{aligned}
\Big\{i\psi^{\chi}_{1,j}(x)\,\pi^{\chi}_{j}(x)\,,\,\psi^{\chi}_{\Lambda,2,j'}(y)^*\,W_{j'}(\varphi^{\chi}_{\Lambda}(y))\Big\} &= \psi^{\chi}_{1,j}(x)\,\psi^{\chi}_{\Lambda,2,j'}(y)^*\,\Big[i\,\pi^{\chi}_{j}(x)\,,\,W_{j'}(\varphi^{\chi}_{\Lambda}(y))\Big] \\
&= \psi^{\chi}_{1,j}(x)\psi^{\chi}_{\Lambda,2,j'}(y)^*\mathcal{K}_{\Lambda,j}(y-x)\,W_{jj'}(\varphi^{\chi}_{\Lambda}(y))\ . \qquad \text{(VII.66)}
\end{aligned}
$$

Here we have used $\left[i\,\pi^\chi_j(x)\,,\,\varphi^\chi_{\Lambda,j'}(y)\right]=\delta_{j,j'}\mathcal{K}_{\Lambda,j}(y-x)$. Note that $\mathcal{K}^f_{\Lambda,1,j}=\mathcal{K}_{\Lambda,j}$. Therefore if we integrate this expression over x and y and sum over j,j', we obtain

$$\sum_{1\le j,j'\le n}\int_0^\ell dx\int_0^\ell dy\left\{i\psi^\chi_{1,j}(x)\,\pi^\chi_j(x)\,,\,\psi^\chi_{\Lambda,2,j'}(y)^*\,W_{j'}(\varphi^\chi_\Lambda(y))\right\}$$
$$=\sum_{1\le j,j'\le n}\int_0^\ell\psi^\chi_{\Lambda,1,j}(y)\,\psi^\chi_{\Lambda,2,j'}(y)^*\,W_{jj'}(\varphi^\chi_\Lambda(y))\,dy\,,\tag{VII.67}$$

which is the first boson-fermion interaction term in (VII.53). An analogous computation yields the third anti-commutator in (VII.61) as the adjoint of (VII.67) and completes the proof that

$$\{D_{\Lambda,1}\,,\,D^*_{\Lambda,1}\}=\sum_{1\le j,j'\le n}\int_0^\ell\int_0^\ell F_{jj'}(x,y)\,dxdy=H_\Lambda+P\,.\tag{VII.68}$$

The second sample calculation that we explain in detail shows $\{D^*_{\Lambda,1}\,,\,D_{\Lambda,2}\}=\phi\widetilde{\mathcal{R}}$, as stated in (VII.59). Again the free terms do not need elaboration, and in this case they give no contribution to the anticommutator. Therefore $\{D^*_{\Lambda,1}\,,\,D_{\Lambda,2}\}=\sum_{1\le j,j'\le n}\int_0^\ell\int_0^\ell G_{jj'}(x,y)\,dxdy$, where

$$\begin{aligned}G_{jj'}(x,y)=&\left\{-i\psi^\chi_{1,j}(x)^*\left(\overline{\pi}^\chi_j(x)-\partial_x\varphi^\chi_j(x)\right)\,,\,\psi^\chi_{\Lambda,1,j'}(y)W_{j'}(\varphi^\chi_\Lambda(y))\right\}e^{-iy\phi/\ell}\\&+\left\{\psi^\chi_{\Lambda,2,j}(x)^*W_j(\varphi^\chi_\Lambda(x))\,,\,i\psi^\chi_{2,j'}(y)\left(\overline{\pi}^\chi_{j'}(y)+\partial_y\varphi^\chi_{j'}(y)\right)\right\}e^{-iy\phi/\ell}\\&+\left\{\psi^\chi_{\Lambda,2,j}(x)^*W_j(\varphi^\chi_\Lambda(x))\,,\,\psi^\chi_{\Lambda,1,j'}(y)W_{j'}(\varphi^\chi_\Lambda(y))\right\}e^{-iy\phi/\ell}\,.\end{aligned}\tag{VII.69}$$

In fact the third anti-commutator in (VII.69) vanishes. Hence,

$$\begin{aligned}G_{jj'}(x,y)\;=\;&-i\delta_{jj'}\mathcal{K}^f_{\Lambda,1,j}(y-x)\left(\overline{\pi}^\chi_j(x)-\partial_x\varphi^\chi_j(x)\right)W_{j'}(\varphi^\chi_\Lambda(y))e^{-iy\phi/\ell}\\&+i\delta_{jj'}\overline{\mathcal{K}^f_{\Lambda,2,j}(x-y)}W_j(\varphi^\chi_\Lambda(x))\left(\overline{\pi}^\chi_{j'}(y)+\partial_y\varphi^\chi_{j'}(y)\right)e^{-iy\phi/\ell}\,.\end{aligned}\tag{VII.70}$$

We use the relations (VII.51) to obtain

$$\overline{\mathcal{K}^f_{\Lambda,2,j}(x-y)}e^{-iy\phi/\ell}=\mathcal{K}^f_{\Lambda,1,j}(x-y)e^{-ix\phi/\ell}=\mathcal{K}_{\Lambda,j}(x-y)e^{-ix\phi/\ell}\,.\tag{VII.71}$$

From this we conclude that after integrating $G_{jj}(x,y)$, the terms proportional to $\overline{\pi^\chi_j}W_j$ cancel, and we obtain

$$\int_0^\ell\int_0^\ell G_{jj}(x,y)\,dxdy=2i\int_0^\ell\int_0^\ell\mathcal{K}_{\Lambda,j}(y-x)\partial_x\varphi^\chi_j(x)W_j(\varphi^\chi_\Lambda(y))e^{-iy\phi/\ell}\,dxdy\,.\tag{VII.72}$$

Since $K_{\Lambda,j}(x-\ell)=e^{-i\Omega_j\phi}K_{\Lambda,j}(x)$, we infer that the bilinear form $K_{\Lambda,j}(y-x)\varphi^\chi_j(x)$ is periodic in x with period ℓ. Thus integration by parts in the variable x gives no end-point contribution, and

$$\int_0^\ell\mathcal{K}_{\Lambda,j}(y-x)\partial_x\varphi^\chi_j(x)\,dx=-\int_0^\ell\left(\partial_x\mathcal{K}_{\Lambda,j}\right)(y-x)\varphi^\chi_j(x)\,dx=\int_0^\ell\left(\partial_y\mathcal{K}_{\Lambda,j}\right)(y-x)\varphi^\chi_j(x)\,dx=\partial_y\varphi^\chi_{\Lambda,j}(y)\,.\tag{VII.73}$$

Insert this in (VII.72), and sum over j. Integrate by parts (this time in the variable y) to obtain

$$\begin{aligned}\left\{D^*_{\Lambda,1}\,,\,D_{\Lambda,2}\right\} &= \sum_{j=1}^{n}\int_0^\ell\int_0^\ell G_{jj}(x,y)\,dxdy = 2i\int_0^\ell\left(\frac{d}{dy}W(\varphi^\chi_\Lambda(y))\right)e^{-iy\phi/\ell}\,dy \\ &= -\frac{2\phi}{\ell}\int_0^\ell W(\varphi^\chi_\Lambda(y))\,e^{-iy\phi/\ell}\,dy = \phi\widetilde{\mathcal{R}}\,, \qquad \text{(VII.74)}\end{aligned}$$

as claimed. Since $W(\varphi^\chi_\Lambda(y))\,e^{-iy\phi/\ell}$ is periodic in the variable y with period ℓ, the end-points give no contribution to the integration by parts. This completes our analysis of Proposition VII.1.3.

VIII Superfields

In this section, we derive the results presented in §VII from the point of view of superfields. Then, we introduce Euclidean superfields and derive the Feynman-Kac formula.

VIII.1 $N = 2$ Superspace

Let $\{e_1,\ldots,e_4\}$ denote the canonical basis of $\mathbb{R}^4$, and $\{\theta^1,\ldots,\theta^4\}$ its image in the complexified exterior algebra $(\Lambda^*\mathbb{R}^4)^{\mathbb{C}} = \Lambda^*\mathbb{R}^4\otimes_{\mathbb{R}}\mathbb{C}$ under the canonical injection. Complex conjugation in $\mathbb{C}$ induces a conjugation on $(\Lambda^*\mathbb{R}^4)^{\mathbb{C}}$. It is convenient to introduce the following generators of the complexified exterior algebra,

$$\begin{aligned}\theta^+ &= \theta^1 + i\theta^2\,, \qquad & \theta^- &= \theta^3 + i\theta^4\,, \\ \overline{\theta}^+ &= \overline{\theta}^1 - i\overline{\theta}^2\,, \qquad & \overline{\theta}^- &= \overline{\theta}^3 - i\overline{\theta}^4\,.\end{aligned}$$

The *space of functions on* $N = 2$ *superspace* is defined by

$$C(\widehat{M}) := C(M)\otimes(\Lambda^*\mathbb{R}^4)^{\mathbb{C}}\,, \qquad \text{(VIII.1)}$$

where M denotes a (possibly compactified) Minkowski space with coordinates t and x. It is useful to introduce light cone coordinates $x^\pm = \frac{1}{2}(t\mp x)$. We shall use the notation

$$\partial_\pm \equiv \frac{\partial}{\partial x^\pm} = \frac{\partial}{\partial t}\mp\frac{\partial}{\partial x}\,. \qquad \text{(VIII.2)}$$

The generators of $N = 2$ supersymmetry on $N = 2$ superspace are the differential operators

$$G_\pm = \tfrac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm\partial_\pm\,, \qquad \text{(VIII.3)}$$

$$\overline{G}_\pm = \tfrac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm\partial_\pm\,, \qquad \text{(VIII.4)}$$

acting on the space of functions on $N = 2$ superspace. The only non-vanishing anti-commutators between the G's are

$$\{G_\pm,\overline{G}_\pm\} = 2i\partial_\pm\,. \qquad \text{(VIII.5)}$$

The identifications

$$H = i\frac{\partial}{\partial t} \ , \quad P = \frac{1}{i}\frac{\partial}{\partial x} \ , \quad D_1 = \frac{1}{\sqrt{2}}G_+ \ , \quad D_2 = \frac{1}{\sqrt{2}}G_- \ , \tag{VIII.6}$$

give a realization of the $N = 2$ algebra (VII.56), (VII.57), and (VII.59) without error terms, i.e., with $\mathcal{R} = \widetilde{\mathcal{R}} = 0$.

The construction of irreducible representations of the $N = 2$ algebra is greatly simplified using the so-called *covariant derivatives*,

$$\nabla_\pm = \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm\partial_\pm \ , \tag{VIII.7}$$

$$\overline{\nabla}_\pm = \frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm\partial_\pm \ . \tag{VIII.8}$$

The covariant derivatives anti-commute with the supercharges $G_\pm$, $\overline{G}_\pm$, and they satisfy the "conjugate" $N = 2$ algebra,

$$\{\nabla_\pm, \overline{\nabla}_\pm\} = -2i\partial_\pm \ , \tag{VIII.9}$$

while all other anti-commutators vanish.

VIII.2 $N = 2$ Chiral Superfields

A general $N = 2$ superfield Φ is (classically) a function on $N = 2$ superspace. Expanding a superfield in powers of the Grassmann coordinates and their complex conjugate, we can express it in terms of 16 fields on two dimensional Minkowski space. However, the resulting representation of the $N = 2$ algebra is highly reducible. One obtains irreducible representations by introducing covariant constraints on the superfield. The constraints we shall be interested in read,

$$\overline{\nabla}_\pm\Phi(x^\pm, \theta^\pm, \overline{\theta}^\pm) = 0 \ , \tag{VIII.10}$$

and they define so-called *chiral superfields*. In order to solve the constraint (VIII.10), we introduce chiral coordinates on $N = 2$ superspace by setting,

$$y^\pm = x^\pm - i\theta^\pm\overline{\theta}^\pm \ . \tag{VIII.11}$$

The chiral coordinates satisfy

$$\overline{\nabla}_\pm y^\pm = 0 \ , \tag{VIII.12}$$

and the most general solution to (VIII.10) is of the form

$$\begin{aligned}\Phi(x^\pm, \theta^\pm, \overline{\theta}^\pm) &= \varphi(y^\pm) + \sqrt{2}\theta^+\psi_1(y^\pm) + \sqrt{2}\theta^-\psi_2^*(y^\pm) + 2\theta^+\theta^-F(y^\pm) \\ &= \varphi(x^\pm) + \sqrt{2}\theta^+\psi_1(x^\pm) + \sqrt{2}\theta^-\psi_2^*(x^\pm) + 2\theta^+\theta^-F(x^\pm) \\ &\quad -\theta^+\overline{\theta}^+ i\partial_+\varphi(x^\pm) - \theta^-\overline{\theta}^- i\partial_-\varphi(x^\pm) + \sqrt{2}\theta^+\theta^-\overline{\theta}^+ i\partial_+\psi_2^*(x^\pm) \\ &\quad -\sqrt{2}\theta^+\theta^-\overline{\theta}^- i\partial_-\psi_1(x^\pm) + \theta^+\theta^-\overline{\theta}^+\overline{\theta}^-\partial_+\partial_-\varphi(x^\pm) \ ,\end{aligned} \tag{VIII.13}$$

where φ, a are complex bosonic- and ψ_1, ψ_2 are complex fermionic fields.

Let ϵ_+, and ϵ_- denote *constant* complex Grassmann parameters, i.e., elements of degree 1 in $(\Lambda^*\mathbb{R}^4)^{\mathbb{C}}$. A general supersymmetry transformation is generated by

$$G = \epsilon_+ G_+ + \epsilon_- G_- + \overline{\epsilon}_+ \overline{G}_+ + \overline{\epsilon}_- \overline{G}_- \ , \qquad \text{(VIII.14)}$$

where as before $\overline{\epsilon}_\pm$ denotes the conjugate element of ϵ_+ in $(\Lambda^*\mathbb{R}^4)^{\mathbb{C}}$. Using (VIII.3), (VIII.4) and (VIII.13), one verifies that under supersymmetry, the component fields transform as follows,

$$\begin{aligned}
\delta\varphi &= \sqrt{2}(\epsilon_+\psi_1 + \epsilon_-\psi_2^*) \ , && \text{(VIII.15)}\\
\delta\psi_1 &= \sqrt{2}(\epsilon_- F - i\overline{\epsilon}_+\partial_+\varphi) \ , && \text{(VIII.16)}\\
\delta\psi_2 &= \sqrt{2}(-\overline{\epsilon}_+ F^* + i\epsilon_-\partial_-\varphi^*) \ , && \text{(VIII.17)}\\
\delta F &= \sqrt{2}i(\overline{\epsilon}_+\partial_+\psi_2^* - \overline{\epsilon}_-\partial_-\psi_1) \ , && \text{(VIII.18)}
\end{aligned}$$

where, for example, $\delta\varphi$ denotes $(G\Phi)_{\theta^\pm=\overline{\theta}^\pm=0}$. Since the covariant derivatives commute with the supersymmetry transformations, chiral superfields are mapped to chiral superfields.

VIII.3 Supersymmetric Lagrangians

Let W be a holomorphic, quasi-homogeneous polynomial with weights $\{\Omega_i\}_{i=1,\ldots,n}$ and with the same properties as described at the beginning of §VII.1. We denote by $\Phi = \{\Phi_i\}_{i=1,\ldots,n}$ a family of chiral superfields. Let us momentarily, for the rest of this section, consider classical fields. The Lagrangian density,

$$\begin{aligned}
\mathcal{L} &= \int d^2\theta d^2\overline{\theta}(-\frac{1}{4}\Phi^*\Phi) + (\int d^2\theta W(\Phi)|_{\overline{\theta}^\pm=0} + \text{h.c.})\\
&= \sum_{i=1}^n(\frac{1}{2}\partial_+\varphi_i^*\partial_-\varphi_i + \frac{1}{2}\partial_-\varphi_i^*\partial_+\varphi_i + i\psi_{1,i}^*\partial_-\psi_{1,i} + i\psi_{2,i}^*\partial_+\psi_{2,i} + F_i^*F_i\\
&\quad +(F_i\partial_i W(\varphi) - \sum_{j=1}^n \psi_{1,i}\psi_{2,j}^*\partial_i\partial_j W(\varphi) + \text{h.c.}) + \text{divergence} \ , \qquad \text{(VIII.19)}
\end{aligned}$$

is invariant under supersymmetry transformations. The so-called *auxiliary fields* F_i are not dynamical and their equations of motion read

$$F_i + \partial_i W(\varphi) = 0 \ . \qquad \text{(VIII.20)}$$

Eliminating the auxiliary fields form the Lagrangian density (VIII.19) using their equations of motion, one obtains

$$\begin{aligned}
\mathcal{L} &= \sum_{i=1}^n(\frac{1}{2}\partial_+\varphi_i^*\partial_-\varphi_i + \frac{1}{2}\partial_-\varphi_i^*\partial_+\varphi_i - |\partial_i W(\varphi)|^2 + i\psi_{1,i}^*\partial_-\psi_{1,i} + i\psi_{2,i}^*\partial_+\psi_{2,i}\\
&\quad -(\sum_{j=1}^n \psi_{1,i}\psi_{2,j}^*\partial_i\partial_j W(\varphi) + \text{h.c.})) \ , \qquad \text{(VIII.21)}
\end{aligned}$$

which is invariant under supersymmetry transformations up to equations of motion. The supercharge densities associated to $N = 2$ supersymmetry are those given in (VII.39) and (VII.40),

$$D_1(x) = \sum_{j=1}^{n}(i\psi_{1,j}(x)\partial_+\varphi_j^*(x) + \psi_{2,j}(x)\overline{\partial_j W(\varphi(x))}) , \qquad \text{(VIII.22)}$$

$$D_2(x) = \sum_{j=1}^{n}(i\psi_{2,j}(x)\partial_-\varphi_j(x) + \psi_{1,j}(x)\partial_j W(\varphi(x))) , \qquad \text{(VIII.23)}$$

as is easily verified using Noether's theorem. Throughout this section, we shall slightly abuse notation and set

$$\pi_j(x) = \frac{\partial}{\partial t}\varphi_j^*(x) , \quad \overline{\pi}_j(x) = \frac{\partial}{\partial t}\varphi_j(x) . \qquad \text{(VIII.24)}$$

VIII.4 Twist Fields

We investigate the case of twist fields. Suppose the spatial coordinate x is compactified on a circle of length ℓ and that the component fields satisfy the twist relations

$$\varphi_j^\chi(x+\ell) = e^{i\Omega_j^b\phi}\varphi_j^\chi(x) , \qquad \text{(VIII.25)}$$

$$\psi_{\alpha,j}^\chi(x+\ell) = e^{i\Omega_{\alpha,j}^f\phi}\psi_{\alpha,j}^\chi(x) , \qquad \text{(VIII.26)}$$

where ϕ is a real parameter. Since the Lagrangian density (VIII.19) must be periodic with period ℓ, we obtain the following relations between the twisting angles and the weights of the superpotential,

$$\Omega_j^b = \Omega_j , \qquad \text{(VIII.27)}$$

$$\Omega_{1j}^f = \Omega_j - (c+\frac{1}{2}) , \quad \Omega_{2j}^f = -\Omega_j - (c-\frac{1}{2}) , \qquad \text{(VIII.28)}$$

for all $j = 1, \ldots, n$, where c is an arbitrary real parameter. Furthermore, the auxiliary field has to satisfy the twist relation

$$F_j^\chi(x+\ell) = e^{i\Omega_j^F\phi}F_j^\chi(x) , \qquad \text{(VIII.29)}$$

with

$$\Omega_j^F = \Omega_j - 1 , \qquad \text{(VIII.30)}$$

for all $j = 1, \ldots, n$. The above relations can be rewritten as a twist relation for the superfields,

$$e^{-i\Omega_j\phi}\Phi_j^\chi(t, x+\ell, e^{i(c+\frac{1}{2})\phi}\theta^+, e^{-i(c-\frac{1}{2})\phi}\theta^-) = \Phi_j^\chi(t, x, \theta^+, \theta^-) . \qquad \text{(VIII.31)}$$

It follows from (VIII.15)–(VIII.18), that supersymmetry transformations are well defined for twist fields only if

$$\epsilon_\pm(x+\ell) = e^{i(c\pm\frac{1}{2})\phi}\epsilon_\pm(x) . \qquad \text{(VIII.32)}$$

This implies that it is not possible to have both ϵ_+ and ϵ_- constant. Thus, in the case of twist fields, $N = 2$ supersymmetry is broken down to either $N = 1$ supersymmetry if we choose $c = \pm 1/2$ or $N = 0$ supersymmetry for other choices of c. In §VII we chose $c = -1/2$ and we saw how supersymmetry was broken down to $N = 1$. The commutation relations of the regularized supercharges $D_{\Lambda,1}$, $D_{\Lambda,2}$ and their adjoints for general values of c are the same as those given in §VII, except for

$$\{D_{\Lambda,1}, D^*_{\Lambda,1}\} = H_\Lambda + P + (c + \frac{1}{2})\phi \mathcal{R}' , \tag{VIII.33}$$

$$\{D_{\Lambda,2}, D^*_{\Lambda,2}\} = H_\Lambda - P - (c - \frac{1}{2})\phi \mathcal{R} , \tag{VIII.34}$$

where

$$\mathcal{R}' = -\frac{2}{l} \sum_{j=1}^{n} \int_0^l : \psi^\chi_{1,j}(x) \psi^{\chi *}_{1,j}(x) : dx . \tag{VIII.35}$$

VIII.5 Euclidean Fields

The construction of Euclidean scalar and Dirac fields is not unique. The Osterwalder-Schrader theory shows that ambiguities in the Euclidean Green's at coinciding times do not influence the quantum fields they determine, see [11]. Furthermore a natural choice of Euclidean Dirac fields involves doubling the number of degrees of freedom, see [12], which we do here. We begin with the free field case $W = 0$, but we still impose the relations (VIII.27), (VIII.28), and (VIII.30). Since there are no interactions, the fields φ^χ_j, $\psi^\chi_{\alpha,j}$ and F^χ_j are independent free massless fields. The *Euclidean fields* φ^E_j, $\overline{\varphi}^E_j$, $\psi^E_{\alpha,j}$ and $\overline{\psi}^E_{\alpha,j}$ are defined on the Euclidean space $[0, \beta] \times [0, \ell]$ and are required to satisfy the following conditions

(E1) The Euclidean fields (anti)commute consistently with their satistics.

(E2) The Euclidean fields act on a Euclidean Fock space, $\mathcal{H}^E$, in such a way that the following correspondences between the imaginary time and the Euclidean sectors define an isomorphism of Gaussian expectations

Imaginary Time	$\longleftrightarrow$	Euclidean
$(t,x) \in [0,\beta] \times [0,\ell]$		$\vec{x} = (t,x) \in [0,\beta] \times [0,\ell]$
$\overline{\varphi}_j^{\chi}(t,x)$		$\overline{\varphi}_j^{E}(\vec{x})$
$\varphi_j^{\chi}(t,x)$		$\varphi_j^{E}(\vec{x})$
$\overline{\psi}_{\alpha,j}^{\chi}(t,x)$		$\overline{\psi}_{\alpha,j}^{E}(\vec{x})$
$\psi_{\alpha,j}^{\chi}(t,x)$		$\psi_{\alpha,j}^{E}(\vec{x})$
$\langle\, (\cdot)_+ \,\rangle_{\mathcal{T}}$		$\langle\, \cdot \,\rangle_0$

where $\langle\, (\cdot)_+ \,\rangle_{\mathcal{T}}$ denotes the time ordered twisted Gibbs expectation defined in §II.3 and §V.5, and $\langle\, \cdot \,\rangle_0$ denotes the vacuum expectation on the Euclidean Fock space. Recall that $\mathcal{T} = \{\Omega, \theta, \sigma, \ell, \beta\}$ specifies the twisting angles, the twisting group element inserted in the Gibbs expectation and the size of space-time. The above correspondence means for example that

$$\left\langle (\overline{\psi}_{\alpha,j}^{\chi}(t,x)\psi_{\beta,k}^{\chi}(s,y))_+ \right\rangle_{\mathcal{T}} = \left\langle \overline{\psi}_{\alpha,j}^{E}(\vec{x})\psi_{\beta,k}^{E}(\vec{y}) \right\rangle_0 \ . \tag{VIII.36}$$

The expressions for the Euclidean fields are not unique and we may choose any convenient representation. In the following we give explicit formulas for these fields.

For each $j = 1, \ldots, n$, we define

$$\begin{aligned} K_j^b &= \frac{2\pi}{\ell}\mathbb{Z} - \frac{\Omega_j \phi}{\ell} \ , \\ K_j^b(k) &= \frac{2\pi}{\beta}\mathbb{Z} - \frac{\Omega_j \phi + \sigma k}{\beta} \ , \quad k \in K_j^b \ , \\ \Lambda_j^b &= \{(E,k) | k \in K_j^b \ , \ E \in K_j^b(k)\} \ . \end{aligned}$$

We construct the Euclidean bosonic Hilbert space $\mathcal{H}_b^E$ in the same way as we did for the real time Hilbert space in section §II.1, i.e.,

$$\mathcal{H}_b^E = \exp_{\otimes_s} \mathcal{K}_b \ , \tag{VIII.37}$$

where

$$\mathcal{K}_b = \bigoplus_{i=1}^{n} (l_2(\Lambda_j^b) \oplus l_2(-\Lambda_j^b)) \ . \tag{VIII.38}$$

We shall denote the creation operators acting on $\mathcal{H}_b^E$ by $A_{\pm,j}^*(\pm\vec{p})$, where $\vec{p} \in \Lambda_j^b$. The Euclidean bosonic fields can be written as

$$\begin{aligned} \varphi_j^E(\vec{x}) &= \frac{1}{\sqrt{\beta\ell}} \sum_{\vec{p} \in \Lambda_j^b} \frac{1}{|\vec{p}|} (A_{+,j}^*(\vec{p}) + A_{-,j}(-\vec{p})) e^{-i\vec{p}\vec{x}} \ , \\ \overline{\varphi}_j^E(\vec{x}) &= (\varphi_j^E(\vec{x}))^* \ . \end{aligned}$$

It is easily verified that the vacuum expectation of two Euclidean bosonic fields reproduces the twisted Gibbs expectation of imaginary time bosonic fields computed in Proposition II.4.2.

Next, we describe the Euclidean Fermi fields. As for the bosonic fields, we define for each $j = 1, \ldots, n$,

$$\begin{aligned} K^f_{\alpha,j} &= \frac{2\pi}{\ell}\mathbb{Z} - \frac{\Omega^f_{\alpha,j}\phi}{\ell} , \\ K^f_{\alpha,j}(k) &= \frac{2\pi}{\beta}\mathbb{Z} - \frac{\Omega^f_{\alpha,j}\phi + \sigma k}{\beta} , \quad k \in K^f_{\alpha,j} , \\ \Lambda^f_{\alpha,j} &= \{(E,k) | k \in K^b_{\alpha,j} , \ E \in K^b_{\alpha,j}(k)\} . \end{aligned}$$

We construct the Euclidean fermionic Hilbert space as in §V.1,

$$\mathcal{H}^E_f = \exp_\wedge \mathcal{K}_f , \tag{VIII.39}$$

where

$$\mathcal{K}_f = \bigoplus_{i=1}^{n} \bigoplus_{\alpha=1,2} (l_2(\Lambda^f_{\alpha,j}) \oplus l_2(-\Lambda^f_{\alpha,j})) . \tag{VIII.40}$$

The creation operators acting on $\mathcal{H}^E_f$ will be denoted by $d^*_{\sigma,j}(\vec{p})$ and $e^*_{\sigma,j}(-\vec{p})$ for $\vec{p} \in \Lambda^f_{\sigma+1,j}$, where $\sigma + 1$ is meant modulo 2.

In order to describe the Euclidean fermionic fields, we need to introduce the following spinors:

For each $\alpha = 1, 2$ and $\vec{p} \in \Lambda^f_{\alpha+1,j}$, we define

$$u^1_j(\vec{p}) = \frac{1}{\sqrt{\beta l(E - ik)}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \quad u^2_j(\vec{p}) = \frac{1}{\sqrt{\beta l(E + ik)}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \tag{VIII.41}$$

$$v^1_{\alpha,j}(-\vec{p}) = u^1_{\alpha,j}(\vec{p}) , \qquad v^2_{\alpha,j}(-\vec{p}) = u^2_{\alpha,j}(\vec{p}) , \tag{VIII.42}$$

and for $\vec{p} \in \Lambda^f_{\alpha,j}$, we set

$$\hat{u}^1_j(\vec{p}) = \frac{1}{\sqrt{\beta l(E - ik)}} \begin{pmatrix} 0 \\ -1 \end{pmatrix} , \quad \hat{u}^2_j(\vec{p}) = \frac{1}{\sqrt{\beta l(E + ik)}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{VIII.43}$$

$$\hat{v}^1_{\alpha,j}(-\vec{p}) = -\hat{u}^1_{\alpha,j}(\vec{p}) , \qquad \hat{v}^2_{\alpha,j}(-\vec{p}) = -\hat{u}^2_{\alpha,j}(\vec{p}) . \tag{VIII.44}$$

We are ready to write down the Euclidean fermionic operators,

$$\overline{\psi}^E_{\alpha,j}(\vec{x}) = \sum_{\substack{\vec{p} \in \Lambda_{\alpha+1} \\ \sigma = 1,2}} (d_{\sigma,j}(\vec{p}) u^\sigma_{\alpha,j}(\vec{p}) + e^*_{\sigma,j}(-\vec{p}) v^\sigma_{\alpha,j}(-\vec{p})) e^{i\vec{p}\vec{x}} \tag{VIII.45}$$

$$\psi^E_{\alpha,j}(\vec{x}) = \sum_{\substack{\vec{p} \in \Lambda_{\alpha} \\ \sigma = 1,2}} (e_{\sigma,j}(-\vec{p}) \hat{v}^\sigma_{\alpha,j}(-\vec{p}) + d^*_{\sigma,j}(\vec{p}) \hat{u}^\sigma_{\alpha,j}(\vec{p})) e^{-i\vec{p}\vec{x}} . \tag{VIII.46}$$

The explicit form of the Euclidean Fermi fields is quite simple,

$$\begin{aligned}
\psi^E_{1,j}(\vec{x}) &= \sum_{\vec{p}\in\Lambda^f_{1,j}} \frac{1}{\sqrt{\beta\ell(E+ik)}}(-e_{2,j}(-\vec{p})+d^*_{2,j}(\vec{p}))e^{-i\vec{p}\vec{x}} ,\\
\psi^E_{2,j}(\vec{x}) &= \sum_{\vec{p}\in\Lambda^f_{2,j}} \frac{1}{\sqrt{\beta\ell(E-ik)}}(e_{1,j}(-\vec{p})-d^*_{1,j}(\vec{p}))e^{-i\vec{p}\vec{x}} ,\\
\overline{\psi}^E_{1,j}(\vec{x}) &= \sum_{\vec{p}\in\Lambda^f_{2,j}} \frac{1}{\sqrt{\beta\ell(E-ik)}}(e^*_{1,j}(-\vec{p})+d_{1,j}(\vec{p}))e^{i\vec{p}\vec{x}} ,\\
\overline{\psi}^E_{2,j}(\vec{x}) &= \sum_{\vec{p}\in\Lambda^f_{1,j}} \frac{1}{\sqrt{\beta\ell(E+ik)}}(e^*_{2,j}(-\vec{p})+d_{2,j}(\vec{p}))e^{i\vec{p}\vec{x}} . \qquad \text{(VIII.47)}
\end{aligned}$$

Using these equations, it is straightforward to check that the requirements (E1) and (E2) are satisfied.

Finally, we introduce the auxiliary Euclidean fields. They are defined to be Gaussian fields with pair correlation functions given as follows:

$$\left\langle (F^\chi_j(t,x)F^\chi_k(s,y))_+ \right\rangle = \left\langle (\overline{F}^\chi_j(t,x)\overline{F}^\chi_k(s,y))_+ \right\rangle = 0 , \qquad \text{(VIII.48)}$$

$$\left\langle (\overline{F}^\chi_j(t,x)F^\chi_k(s,y))_+ \right\rangle = \delta_{j,k}\delta(t-s)\delta(x-y) . \qquad \text{(VIII.49)}$$

It is thus straightforward to write down the Euclidean auxiliary fields, the only subtlety being the twist relations satisfied by these fields. As above, we introduce

$$K^F_j = \frac{2\pi}{\ell}\mathbb{Z} - \frac{\Omega^F_j\phi}{\ell} , \qquad \text{(VIII.50)}$$

$$K^F_j(k) = \frac{2\pi}{\beta}\mathbb{Z} - \frac{\Omega^F_j\phi+\sigma k}{\beta} , \quad k\in K^F_j , \qquad \text{(VIII.51)}$$

$$\Lambda^F_j = \{(E,k)|k\in K^F_j,\ E\in K^F_j(k)\} . \qquad \text{(VIII.52)}$$

The auxiliary Euclidean Hilbert space is given by

$$\mathcal{H}^E_F = \exp_{\otimes_s}\mathcal{K}^F , \qquad \text{(VIII.53)}$$

where

$$\mathcal{K}^F = \bigoplus_{j=1}^{n}(l_2(K^F_j)\oplus l_2(-K^F_j)) . \qquad \text{(VIII.54)}$$

We denote the creation operators by $f^*_{\pm,j}(\pm\vec{p})$, where $\vec{p}\in\Lambda^F_j$. The auxiliary Euclidean fields can then be written as

$$F^E_j(\vec{x}) = \frac{1}{\sqrt{\beta\ell}}\sum_{\vec{p}\in\Lambda^F_j}(f^*_{+,j}(\vec{p})+f_{-,j}(-\vec{p}))e^{-i\vec{p}\vec{x}} \qquad \text{(VIII.55)}$$

$$\overline{F}^E_j(\vec{x}) = (F^E_j(\vec{x}))^* . \qquad \text{(VIII.56)}$$

This finishes our descritpion of the Euclidean fields. They all act on the Euclidean Hilbert space,

$$\mathcal{H}^E = \mathcal{H}_b^E \otimes \mathcal{H}_f^E \otimes \mathcal{H}_F^E \ . \tag{VIII.57}$$

In the next section, we shall use the regularized Euclidean auxiliary fields,

$$F^E_{\Lambda,j}(\vec{x}) \ = \ \frac{1}{\sqrt{\beta\ell}} \sum_{\vec{p}\in\Lambda_j^F} (f^*_{+,j}(\vec{p}) + f_{-,j}(-\vec{p}))\hat{(\mathcal{K})}((k+\Omega_j^F \frac{\phi}{\ell})/\Lambda)e^{-i\vec{p}\vec{x}} \ , \tag{VIII.58}$$

$$\overline{F^E_{\Lambda,j}}(\vec{x}) \ = \ (F^E_{\Lambda,j}(\vec{x}))^* \ . \tag{VIII.59}$$

These fields satisfy,

$$\left\langle F^E_{\Lambda,j}(\vec{x}) F^E_{\Lambda,k}(\vec{y})\right\rangle_0 = \left\langle \overline{F}^E_{\Lambda,j}(\vec{x}) \overline{F}^E_{\Lambda,k}(\vec{y})\right\rangle_0 = 0 \ , \tag{VIII.60}$$

$$\left\langle \overline{F}^E_{\Lambda,j}(\vec{x}) F^E_{\Lambda,k}(\vec{y})\right\rangle_0 = \left\langle F^E_{\Lambda,k}(\vec{y}) \overline{F}^E_{\Lambda,j}(\vec{x})\right\rangle_0 = \delta_{j,k}\delta(t-s)v_{\Lambda,j}(x-y) \ . \tag{VIII.61}$$

VIII.6 The Feyman-Kac formula

Having introduced the Euclidean fields, we describe the Feynamn-Kac identity in the superfield formalism. First, we define the Euclidean chiral superfields, $\Phi^E = \{\Phi^E_j\}$, and their "conjugate",

$$\Phi^E_j(x^\pm,\theta^\pm,\overline{\theta}^\pm) \ = \ \varphi^E_j(y^\pm) + \sqrt{2}\theta^+\psi^E_{1,j}(y^\pm) + \sqrt{2}\theta^-\overline{\psi}^E_{1,j}(y^\pm) + 2\theta^+\theta^- F_j(y^\pm) \ , \tag{VIII.62}$$

$$\overline{\Phi}^E_j(x^\pm,\theta^\pm,\overline{\theta}^\pm) \ = \ \overline{\varphi}^E_j(y^\pm) + \sqrt{2}\overline{\theta}^+\overline{\psi}^E_{2,j}(y^\pm) + \sqrt{2}\overline{\theta}^-\psi^E_{2,j}(y^\pm) + 2\overline{\theta}^+\overline{\theta}^-\overline{F}_j(y^\pm) \ . \tag{VIII.63}$$

The regularized Euclidean action density reads

$$\mathcal{L}^E_\Lambda \ = \ \frac{1}{2}\int d^2\theta W(\Phi^E_\Lambda)|_{\overline{\theta}^\pm=0} + \frac{1}{2}\int d^2\overline{\theta}\,\overline{W}(\overline{\Phi}^E_\Lambda)|_{\theta^\pm=0} \tag{VIII.64}$$

$$\begin{aligned} &= \ \sum_{j=1}^n (\partial_j W(\varphi^E_\Lambda)F_{\Lambda,j} + \overline{\partial}_j\overline{W}(\overline{\varphi}^E_\Lambda)\overline{F}_{\Lambda,j} \\ &\qquad - \sum_{k=1}^n (\psi^E_{\Lambda,1,j}\overline{\psi}^E_{\Lambda,1,k}\partial_j\partial_k W(\varphi^E_\Lambda) - \psi^E_{\Lambda,2,j}\overline{\psi}^E_{\Lambda,2,k}\overline{\partial}_j\overline{\partial}_k\overline{W}(\overline{\varphi}\Lambda^E))) \ . \end{aligned} \tag{VIII.65}$$

The covariance for the regularized Euclidean auxiliary fields $F^E_{\Lambda,j}$ together with the isomorphism of Gaussian expectations realized by the Euclidean fields lead to the Feynman-Kac formula linking the imaginary time to the Euclidean sector

$$\left\langle (\cdot e^{-S_I})_+ \right\rangle_{\mathcal{T}} = \left\langle \cdot e^{-iS^E} \right\rangle_0 \ , \tag{VIII.66}$$

where the interaction action S_I is given by

$$\begin{aligned} S_I = &\sum_{j=1}^n \int_0^\ell dx \int_0^\beta dt [\int_0^l dy \overline{W_j(\varphi^\chi_{\Lambda,j}(t,x))}\ v_{\Lambda,j}(x-y)\ W_j(\varphi^\chi_{\Lambda,j}(t,y)) \\ &+ \sum_{k=1}^n (\psi^\chi_{\Lambda,1,j}(t,x)\overline{\psi}^\chi_{\Lambda,2,k}(t,x)\partial_j\partial_k W(\varphi^\chi_\Lambda(t,x)) + \psi^\chi_{\Lambda,2,j}(t,x)\overline{\psi}^\chi_{\Lambda,1,k}(t,x)\overline{\partial}_j\overline{\partial}_k\overline{W}(\overline{\varphi}^\chi_\Lambda(t,x)))] \ , \end{aligned}$$

and the Euclidean action is given by

$$S^E = \int_0^\beta dx_1 \int_0^\ell dx_2\, \mathcal{L}^E\ . \tag{VIII.67}$$

The proof of (VIII.66) goes as follows for one auxiliary field,

$$\begin{aligned}
&\left\langle \cdot\ e^{-i\int d^2x \sum_{j=1}^n (F^E_{\Lambda,j}(\vec{x})\partial_j W(\varphi^E_\Lambda(\vec{x})) + \overline{F}^E_{\Lambda,j}(\vec{x})\overline{\partial}_j \overline{W}(\overline{\varphi}^E_\Lambda(\vec{x})))} \right\rangle_0 = \\
&= \sum_{k=0}^\infty \frac{(-i)^k}{k!} \left\langle \cdot\ \prod_{l=1}^k \int d^2x_l (F^E_{\Lambda,j}(\vec{x}_l)\partial_j W(\varphi^E_\Lambda(\vec{x}_l)) + \overline{F}^E_{\Lambda,j}(\vec{x}_l)\overline{\partial}_j \overline{W}(\overline{\varphi}^E_\Lambda(\vec{x}_l))) \right\rangle_0 \\
&= \sum_{k=0}^\infty \frac{(-i)^{2k}}{(2k)!}\frac{(2k)!}{k!} \left\langle \cdot\ \prod_{l=1}^k \int_0^\beta dt_l \int_0^\ell dx_l \int_0^\ell dy_l\, \overline{W_j(\varphi^E_{\Lambda,j}(t_l,x_l))}\, v_{\Lambda,j}(x_l - y_l)\, W_j(\varphi^E_{\Lambda,j}(t_l,y_l)) \right\rangle_0 \\
&= \left\langle \cdot\ e^{-\int_0^\beta dt \int_0^\ell dx \int_0^\ell dy\, \overline{W_j(\varphi^E_{\Lambda,j}(t,x))}\ v_{\Lambda,j}(x-y)\ W_j(\varphi^E_{\Lambda,j}(t,y))} \right\rangle_0 \\
&= \left\langle (\cdot\ e^{-\int_0^\beta dt \int_0^\ell dx \int_0^\ell dy\, \overline{W_j(\varphi^\chi_{\Lambda,j}(t,x))}\ v_{\Lambda,j}(x-y)\ W_j(\varphi^\chi_{\Lambda,j}(t,y))})_+ \right\rangle_{\mathcal{T}}
\end{aligned}$$

where we used (VIII.49) and condition (E2) in the second and fourth steps, respectively.

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