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Quantum Field Theory on Curved Backgrounds. II. Spacetime Symmetries

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Abstract We study space-time symmetries in scalar quantum field theory on an arbitrary static space-time. We first consider Euclidean quantum field theory, and show that the isometry group is generated by one-parameter subgroups which have either self-adjoint or unitary quantizations. We then show that the self-adjoint semigroups thus constructed can be analytically continued to one-parameter unitary groups, and using this analytic continuation we construct a unitary representation of the isometry group of the Lorentz-signature metric. We illustrate our method for the explicit example of hyperbolic space, whose Lorentzian continuation is Anti-de Sitter space.

1 Introduction

The extension of quantum field theory to curved space-times has led to the discovery of many qualitatively new phenomena which do not occur in the simpler theory on Minkowski space, such as Hawking radiation; for background and historical references, see [2, 6, 19].

The reconstruction of quantum field theory on a Lorentz-signature space-time from the corresponding Euclidean quantum field theory makes use of Osterwalder-Schrader (OS) positivity [16, 17] and analytic continuation. On a curved background, there may be no proper definition of time-translation and no Hamiltonian; thus, the mathematical framework of Euclidean quantum field theory may break down. However, on static space-times there is a Hamiltonian and it makes sense to define Euclidean QFT. This approach was recently taken by the authors [12], in which the fundamental properties of Osterwalder-Schrader quantization and some of the fundamental estimates of constructive quantum field theory[1] were generalized to static space-times.

The previous work [12], however, did not address the analytic continuation which leads from a Euclidean theory to a real-time theory. In the present article, we initiate a

[1] For background on constructive field theory in flat space-times, see [8, 10].

treatment of the analytic continuation by constructing unitary operators which form a representation of the isometry group of the Lorentz-signature space-time associated to a static Riemannian space-time. Our approach is similar in spirit to that of Fröhlich [4] and of Klein and Landau [14], who showed how to go from the Euclidean group to the Poincaré group without using the field operators on flat space-time.

This work also has applications to representation theory, as it provides a natural (functorial) quantization procedure which constructs nontrivial unitary representations of Lie groups which arise as isometry groups of static, Lorentz-signature space-times. For example, when applied to AdS_{d+1}, our procedure gives a unitary representation of $SO(d,2)$. Although very different in the details, this is reminiscent of the construction of unitary representations through geometric quantization (see for instance [9, 20]).

2 Classical Space-Time

2.1 Structure of Static Space-Times

Definition 2.1 *A* **quantizable static space-time** *is a complete, connected orientable Riemannian manifold (M, g_{ab}) with a globally-defined (smooth) Killing field ξ which is orthogonal to a codimension-one hypersurface $\Sigma \subset M$, such that the orbits of ξ are complete and each orbit intersects Σ exactly once.*

Throughout this paper, we assume that M is a quantizable static space-time. Definition 2.1 implies that there is a global time function t defined up to a constant by the requirement that $\xi = \partial/\partial t$. Thus M is foliated by time-slices M_t, and

$$M = \Omega_- \cup \Sigma \cup \Omega_+$$

where the unions are disjoint, $\Sigma = M_0$, and $\Omega_\pm$ are open sets corresponding to $t > 0$ and $t < 0$ respectively. We infer existence of an isometry θ which reverses the sign of t,

$$\theta : \Omega_\pm \to \Omega_\mp \ \text{ such that } \ \theta^2 = 1, \ \ \theta|_\Sigma = \text{id}.$$

Let $C = (-\Delta + m^2)^{-1}$ be the resolvent of the Laplacian, also called the *free covariance*, where $m^2 > 0$. Then C is a bounded self-adjoint operator on $L^2(M)$. For each $s \in \mathbb{R}$, the Sobolev space $H_s(M)$ is a real Hilbert space, defined as completion of $C_c^\infty(M)$ in the norm

$$\|f\|_s^2 = \langle f, C^{-s} f\rangle. \tag{2.1}$$

The inclusion $H_s \hookrightarrow H_{s+k}$ for $k > 0$ is Hilbert-Schmidt. Define $\mathscr{S} := \bigcap_{s<0} H_s(M)$ and $\mathscr{S}' := \bigcup_{s>0} H_s(M)$. Then

$$\mathscr{S} \ \subset \ H_{-1}(M) \ \subset \ \mathscr{S}'$$

form a Gelfand triple, and $\mathscr{S}$ is a nuclear space.

Recall that $\mathscr{S}'$ has a natural σ-algebra of measurable sets, called *cylinder sets* (see for instance [7, 8, 18]). There is a unique Gaussian probability measure μ with mean zero and covariance C defined on the cylinder sets in $\mathscr{S}'$ (see [7]).

More generally, one may consider a non-Gaussian, countably-additive measure μ on $\mathscr{S}'$ and the space

$$\mathscr{E} \ := \ L^2(\mathscr{S}', \mu).$$

We are interested in the case that the monomials of the form $A(\Phi) = \Phi(f_1)\dots\Phi(f_n)$ are all elements of $\mathscr{E}$, and for which their span is dense in $\mathscr{E}$. For an open set $\Omega \subset M$, let $\mathscr{E}_\Omega$ denote the closure in $\mathscr{E}$ of the set of monomials $A(\Phi) = \prod_i \Phi(f_i)$ where $\mathrm{supp}(f_i) \subset \Omega$ for all i. Of particular importance for Euclidean quantum field theory is the positive-time subspace

$$\mathscr{E}_+ \;:=\; \mathscr{E}_{\Omega_+}\,.$$

2.2 The Operator Induced by an Isometry

Isometries of the underlying space-time manifold act on a Hilbert space of classical fields arising in the study of a classical field theory. For $f \in C^\infty(M)$ and $\psi : M \to M$ an isometry, define

$$f^\psi \;\equiv\; (\psi^{-1})^* f = f \circ \psi^{-1}.$$

Since $\det(d\psi) = 1$, the operation $f \to f^\psi$ extends to a bounded operator on $H_{\pm 1}(M)$ or on $L^2(M)$. An extensive treatment of isometries for static space-times appears in [12].

Definition 2.2 *Let ψ be an isometry, and $A(\Phi) = \Phi(f_1)\dots\Phi(f_n) \in \mathscr{E}$ a monomial. Define the induced operator*

$$\Gamma(\psi)A \;\equiv\; \Phi({f_1}^\psi)\dots\Phi({f_n}^\psi)\,, \tag{2.2}$$

and extend $\Gamma(\psi)$ by linearity to the dense domain of polynomials in $\mathscr{E}$.

3 Osterwalder-Schrader Quantization

3.1 Quantization of Vectors (The Hilbert Space $\mathscr{H}$ of Quantum Theory)

In this section we define the quantization map $\mathscr{E}_+ \to \mathscr{H}$, where $\mathscr{H}$ is the Hilbert space of quantum theory. The existence of the quantization map relies on a condition known as Osterwalder-Schrader (or reflection) positivity. A probability measure μ on cylinder sets in $\mathscr{S}'$ is said to be *reflection positive* if

$$\int \overline{\Gamma(\theta)F}\, F \; d\mu \geq 0 \tag{3.1}$$

for all F in the positive-time subspace $\mathscr{E}_+ \subset \mathscr{E}$. Let $\Theta = \Gamma(\theta)$ be the reflection on $\mathscr{E}$ induced by θ. Define the sesquilinear form (A,B) on $\mathscr{E}_+ \times \mathscr{E}_+$ as $(A,B) = \langle \Theta A, B\rangle_{\mathscr{E}}$.

Assumption 1 (O-S Positivity) *Any measure $d\mu$ that we consider is reflection positive with respect to the time-reflection Θ.*

Definition 3.1 (OS-Quantization) *Given a reflection-positive measure $d\mu$, the Hilbert space $\mathscr{H}$ of quantum theory is the completion of $\mathscr{E}_+/\mathscr{N}$ with respect to the inner product given by the sesquilinear form (A,B). Denote the quantization map Π for vectors $\mathscr{E}_+ \to \mathscr{H}$ by $\Pi(A) = \hat{A}$, and write*

$$\langle \hat{A}, \hat{B}\rangle_{\mathscr{H}} = (A,B) = \langle \Theta A, B\rangle_{\mathscr{E}} \quad \textit{for} \quad A, B \in \mathscr{E}_+\,. \tag{3.2}$$

3.2 Quantization of Operators

The basic quantization theorem gives a sufficient condition to map a (possibly unbounded) linear operator T on $\mathscr{E}$ to its quantization, a linear operator $\hat{T}$ on $\mathscr{H}$. Consider a densely-defined operator T on $\mathscr{E}$, the unitary time-reflection Θ, and the adjoint $T^+ = \Theta T^* \Theta$. A preliminary version of the following was also given in [11].

Definition 3.2 (Quantization Condition I) *The operator T satisfies QC-I if:*

i. The operator T has a domain $\mathscr{D}(T)$ dense in $\mathscr{E}$.
ii. There is a subdomain $\mathscr{D}_0 \subset \mathscr{E}_+ \cap \mathscr{D}(T) \cap \mathscr{D}(T^+)$, for which $\widehat{\mathscr{D}}_0 \subset \mathscr{H}$ is dense.
iii. The transformations T and T^+ both map $\mathscr{D}_0$ into $\mathscr{E}_+$.

Theorem 3.1 (Quantization I) *If T satisfies QC-I, then*

i. The operators $T \restriction \mathscr{D}_0$ and $T^+ \restriction \mathscr{D}_0$ have quantizations $\hat{T}$ and $\widehat{T^+}$ with domain $\hat{\mathscr{D}}_0$.
ii. The operators $\hat{T}^* = \left(\hat{T} \restriction \hat{\mathscr{D}}_0\right)^*$ and $\widehat{T^+}$ agree on $\hat{\mathscr{D}}_0$.
iii. The operator $\hat{T} \restriction \mathscr{D}_0$ has a closure, namely $\hat{T}^{**}$.

Proof We wish to define the quantization $\hat{T}$ with the putative domain $\hat{\mathscr{D}}_0$ by

$$\hat{T}\hat{A} = \widehat{TA} \; . \tag{3.3}$$

For any vector $A \in \mathscr{D}_0$ and for any $B \in (\mathscr{D}_0 \cap \mathscr{N})$, it is the case that $\hat{A} = \widehat{A+B}$. The transformation $\hat{T}$ is defined by (3.3) iff $\widehat{TA} = T\widehat{(A+B)} = \widehat{TA} + \widehat{TB}$. Hence one needs to verify that $T : \mathscr{D}_0 \cap \mathscr{N} \to \mathscr{N}$, which we now do.

The assumption $\mathscr{D}_0 \subset \mathscr{D}(T^+)$, along with the fact that Θ is unitary, ensures that $\Theta \mathscr{D}_0 \subset \mathscr{D}(T^*)$. Therefore for any $F \in \mathscr{D}_0$,

$$\langle \Theta F, TB \rangle_{\mathscr{E}} = \langle T^* \Theta F, B \rangle_{\mathscr{E}} = \langle \Theta \left(\Theta T^* \Theta F \right), B \rangle_{\mathscr{E}} \; = \langle \Theta T^+ F, B \rangle_{\mathscr{E}} \; = \langle \widehat{T^+ F}, \hat{B} \rangle_{\mathscr{H}} \; . \tag{3.4}$$

In the last step we use the fact assumed in QC-I.iii that $T^+ : \mathscr{D}_0 \to \mathscr{E}_+$, yielding the inner product of two vectors in $\mathscr{H}$. We infer from the Schwarz inequality in $\mathscr{H}$ that

$$|\langle \Theta F, TB \rangle_{\mathscr{E}}| \leq \|\widehat{T^+ F}\|_{\mathscr{H}} \, \|\hat{B}\|_{\mathscr{H}} = 0 \; .$$

As $\langle \Theta F, TB \rangle_{\mathscr{E}} = \langle \hat{F}, \widehat{TB} \rangle_{\mathscr{H}}$, this means that $\widehat{TB} \perp \hat{\mathscr{D}}_0$. As $\hat{\mathscr{D}}_0$ is dense in $\mathscr{H}$ by QC-I.ii, we infer $\widehat{TB} = 0$. In other words, $TB \in \mathscr{N}$ as required to define $\hat{T}$.

In order show that $\hat{\mathscr{D}}_0 \subset \mathscr{D}(\hat{T}^*)$, perform a similar calculation to (3.4) with arbitrary $A \in \mathscr{D}_0$ replacing B, namely

$$\langle \hat{F}, \hat{T}\hat{A} \rangle_{\mathscr{H}} = \langle \Theta F, TA \rangle_{\mathscr{E}} = \langle \Theta \left(\Theta T^* \Theta F \right), A \rangle_{\mathscr{E}} \; = \langle \Theta T^+ F, A \rangle_{\mathscr{E}} \; = \langle \widehat{T^+ F}, \hat{A} \rangle_{\mathscr{H}} \; . \tag{3.5}$$

The right side is continuous in $\hat{A} \in \mathscr{H}$, and therefore $\hat{F} \in \mathscr{D}(T^*)$. Furthermore $T^* \hat{F} = \widehat{T^+ F}$. This identity shows that if $F \in \mathscr{N}$, then $\widehat{T^+ F} = 0$. Hence $T^+ \restriction \mathscr{D}_0$ has a quantization $\widehat{T^+}$, and we may write (3.5) as

$$T^* \hat{F} = \widehat{T^+} \hat{F} \; , \quad \text{for all } \; F \in \mathscr{D}_0 \; . \tag{3.6}$$

In particular $\hat{T}^*$ is densely defined so $\hat{T}$ has a closure. This completes the proof.

Definition 3.3 (Quantization Condition II) *The operator T satisfies QC-II if*

i. Both the operator T and its adjoint T^* have dense domains $\mathscr{D}(T), \mathscr{D}(T^*) \subset \mathscr{E}$.
ii. There is a domain $\mathscr{D}_0 \subset \mathscr{E}_+$ in the common domain of T, T^+, T^+T, and TT^+.
iii. Each operator T, T^+, T^+T, and TT^+ maps $\mathscr{D}_0$ into $\mathscr{E}_+$.

Theorem 3.2 (Quantization II) *If T satisfies QC-II, then*

i. The operators $T \restriction \mathscr{D}_0$ and $T^+ \restriction \mathscr{D}_0$ have quantizations $\hat{T}$ and $\widehat{T^+}$ with domain $\hat{\mathscr{D}}_0$.
ii. If $A, B \in \mathscr{D}_0$, one has $\langle \hat{B}, \hat{T}\hat{A}\rangle_{\mathscr{H}} = \langle \widehat{T^+}\hat{B}, \hat{A}\rangle_{\mathscr{H}}$.

Remarks.
i. In Theorem 3.2 we drop the assumption that the domain $\hat{\mathscr{D}}_0$ is dense, obtaining quantizations $\hat{T}$ and $\widehat{T^+}$ whose domains are not necessarily dense. In order to compensate for this, we assume more properties concerning the domain and the range of T^+ on $\mathscr{E}$.
ii. As $\hat{\mathscr{D}}_0$ need not be dense in $\mathscr{H}$, the adjoint of $\hat{T}$ need not be defined. Nevertheless, one calls the operator $\hat{T}$ *symmetric* in case one has

$$\langle \hat{B}, \hat{T}\hat{A}\rangle_{\mathscr{H}} = \langle \hat{T}\hat{B}, \hat{A}\rangle_{\mathscr{H}} \ , \qquad \text{for all } \ A, B \in \mathscr{D}_0 \ . \tag{3.7}$$

iii. If $\hat{S} \supset \hat{T}$ is a densely-defined extension of $\hat{T}$, then $\hat{S}^* = \widehat{T^+}$ on the domain $\hat{\mathscr{D}}_0$.

Proof We define the quantization $\hat{T}$ with the putative domain $\hat{\mathscr{D}}_0$. As in the proof of Theorem 3.1, this quantization $\hat{T}$ is well-defined iff it is the case that $T : \mathscr{D}_0 \cap \mathscr{N} \to \mathscr{N}$. For any $F \in \mathscr{D}_0 \cap \mathscr{N}$, by definition $\|\hat{F}\|_{\mathscr{H}} = 0$. Also

$$\langle TF, TF\rangle_{\mathscr{H}} = (TF, TF) = \langle \Theta TF, TF\rangle_{\mathscr{E}} = \langle F, T^*\Theta TF\rangle_{\mathscr{E}} \ ,$$

where one uses the fact that $\mathscr{D}_0 \subset \mathscr{D}(T^+T)$. Thus

$$\langle TF, TF\rangle_{\mathscr{H}} = \left\langle \Theta F, T^+TF\right\rangle_{\mathscr{E}} = \langle F, T^+TF\rangle_{\mathscr{H}} \ .$$

Here we use the fact that T^+T maps $\mathscr{D}_0$ to $\mathscr{E}_+$. Thus one can use the Schwarz inequality on $\mathscr{H}$ to obtain

$$\langle TF, TF\rangle_{\mathscr{H}} \leq \|\hat{F}\|_{\mathscr{H}} \, \|\widehat{T^+TF}\|_{\mathscr{H}} = 0 \ .$$

Hence $T : \mathscr{D}_0 \cap \mathscr{N} \to \mathscr{N}$, and T has a quantization $\hat{T}$ with domain $\hat{\mathscr{D}}_0$.

In order verify that $T^+ \restriction \mathscr{D}_0$ has a quantization, one needs to show that $T^+ : \mathscr{D}_0 \cap \mathscr{N} \subset \mathscr{N}$. Repeat the argument above with T^+ replacing T. The assumption $TT^+ : \mathscr{D}_0 \to \mathscr{E}_+$ yields for $F \in \mathscr{D}_0 \cap \mathscr{N}$,

$$\langle T^+F, T^+F\rangle_{\mathscr{H}} = \langle T^*\Theta F, T^+F\rangle_{\mathscr{E}} = \langle \Theta F, TT^+F\rangle_{\mathscr{E}} = \langle \hat{F}, \widehat{TT^+F}\rangle_{\mathscr{H}} \ .$$

Use the Schwarz inequality in $\mathscr{H}$ to obtain the desired result that

$$\langle T^+F, T^+F\rangle_{\mathscr{H}} \leq \|\hat{F}\|_{\mathscr{H}} \|\widehat{TT^+F}\|_{\mathscr{H}} = 0 \ .$$

Hence T^+ has a quantization $\widehat{T^+}$ with domain $\hat{\mathscr{D}}_0$, and for $B \in \mathscr{D}_0$ one has $\widehat{T^+B} = \widehat{T^+}\hat{B}$. In order to establish (ii), assume that $A, B \in \mathscr{D}_0$. Then

$$\begin{aligned} \langle \hat{B}, \hat{T}\hat{A}\rangle_{\mathscr{H}} &= \langle \Theta B, TA\rangle_{\mathscr{E}} = \langle \Theta\left(\Theta T^*\Theta B\right), A\rangle_{\mathscr{E}} \ = \langle \Theta T^+B, A\rangle_{\mathscr{E}} \\ &= \langle \widehat{T^+B}, \hat{A}\rangle_{\mathscr{H}} = \langle \widehat{T^+}\hat{B}, \hat{A}\rangle_{\mathscr{H}} \ . \end{aligned} \tag{3.8}$$

This completes the proof.

3.3 Applications

The case of Euclidean symmetry for $(t,\mathbf{x}) \in M = \mathbb{R}^d$ was treated by Fröhlich [4] and Klein and Landau [14]. The generalization to arbitrary static, real-analytic space-times is given in the following sections.

4 Structure of the Lie Algebra of Killing Fields

For the remainder of this paper we assume the following, which is clearly true in the Gaussian case as the Laplacian commutes with the isometry group G. (See also [12].)

Assumption 2 *The isometry groups G that we consider leave the measure $d\mu$ invariant, in the sense that G has a unitary representation on $\mathscr{E}$.*

4.1 The Representation of $\mathfrak{g}$ on $\mathscr{E}$

Lemma 4.1 *Let G_i be an analytic group with Lie algebra $\mathfrak{g}_i$ ($i = 1,2$), and let $\lambda : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a homomorphism. There cannot exist more than one analytic homomorphism $\pi : G_1 \to G_2$ for which $d\pi = \lambda$. If G_1 is simply connected then there is always one such π.*

Let $D = d/dt$ denote the canonical unit vector field on $\mathbb{R}$. Let G be a real Lie group with algebra $\mathfrak{g}$, and let $X \in \mathfrak{g}$. The map $tD \to tX (t \in \mathbb{R})$ is a homomorphism of $\mathrm{Lie}(\mathbb{R}) \to \mathfrak{g}$, so by the Lemma there is a unique analytic homomorphism $\xi_X : R \to G$ such that $d\xi_X(D) = X$. Conversely, if η is an analytic homomorphism of $\mathbb{R} \to G$, and if we let $X = d\eta(D)$, it is obvious that $\eta = \xi_X$. Thus $X \mapsto \xi_X$ is a bijection of $\mathfrak{g}$ onto the set of analytic homomorphisms $\mathbb{R} \to G$. The exponential map is defined by $\exp(X) := \xi_X(1)$. For complex Lie groups, the same argument applies, replacing $\mathbb{R}$ with $\mathbb{C}$ throughout.

Since $\mathfrak{g}$ is connected, so is $\exp(\mathfrak{g})$. Hence $\exp(\mathfrak{g}) \subseteq G^0$, where G^0 denotes the connected component of the identity in G. It need not be the case for a general Lie group that $\exp(\mathfrak{g}) = G^0$, but for a large class of examples (the so-called *exponential groups*) this does hold. For any Lie group, $\exp(\mathfrak{g})$ contains an open neighborhood of the identity, so the subgroup generated by $\exp(\mathfrak{g})$ always coincides with G^0.

We will apply the above results with $G = \mathrm{Iso}(M)$, the isometry group of M, and $\mathfrak{g} = \mathrm{Lie}(G)$ the algebra of global Killing fields. Thus we have a bijective correspondence between Killing fields and 1-parameter groups of isometries. This correspondence has a geometric realization: the 1-parameter group of isometries

$$\phi_s = \xi_X(s) = \exp(sX)$$

corresponding to $X \in \mathfrak{g}$ is the flow generated by X.

Consider the two different 1-parameter groups of unitary operators:

1. the unitary group ϕ_s^* on $L^2(M)$, and
2. the unitary group $\Gamma(\phi_s)$ on $\mathscr{E}$.

Stone's theorem applies to both of these unitary groups to yield densely-defined self-adjoint operators on the respective Hilbert spaces.

In the first case, the relevant self-adjoint operator is simply an extension of $-iX$, viewed as a differential operator on $C_c^\infty(M)$. This is because for $f \in C_c^\infty(M)$ and $p \in M$, we have:

$$X_p f = (\mathscr{L}_X f)(p) = \frac{d}{ds} f(\phi_s(p))|_{s=0}.$$

Thus $-iX$ is a densely-defined symmetric operator on $L^2(M)$, and Stone's theorem implies that $-iX$ has self-adjoint extensions.

In the second case, the unitary group $\Gamma(\phi_s)$ on $\mathscr{E}$ also has a self-adjoint generator $\Gamma(X)$, which can be calculated explicitly. By definition,

$$e^{-is\Gamma(X)}\Big[\prod_{i=1}^n \Phi(f_i)\Big] = \prod_{i=1}^n \Phi(f_i \circ \phi_{-s}).$$

Now replace $s \to -s$ and calculate $d/ds|_{s=0}$ applied to both sides of the last equation to see that

$$\Gamma(X)\Big[\prod_{i=1}^n \Phi(f_i)\Big] = \sum_{j=1}^n \Phi(f_1)\ldots\Phi(-iXf_j)\Phi(f_{j+1})\ldots\Phi(f_n)\,.$$

One may check that Γ is a Lie algebra representation of $\mathfrak{g}$, i.e. $\Gamma([X,Y]) = [\Gamma(X),\Gamma(Y)]$.

4.2 A Direct Sum Decomposition of $\mathfrak{g}$

For each $\xi \in \mathfrak{g}$, there exists some dense domain in $\mathscr{E}$ on which $\Gamma(\xi)$ is self-adjoint, as discussed previously. However, the quantizations $\widehat{\Gamma}(\xi)$ acting on $\mathscr{H}$ may be hermitian, anti-hermitian, or neither depending on whether there holds a relation of the form

$$\Gamma(\xi)\Theta = \pm\Theta\Gamma(\xi), \tag{4.1}$$

with one of the two possible signs, or whether no such relation holds.

Even if (4.1) holds, to complete the construction of a unitary representation one must prove that there exists a dense domain in $\mathscr{H}$ on which the quantization $\hat{\xi}$ is self-adjoint or skew-adjoint. This nontrivial problem will be dealt with in a later section using Theorems 3.1 and 3.2 and the theory of symmetric local semigroups [13, 4]. Presently we determine *which* elements within $\mathfrak{g}$ satisfy relations of the form (4.1).

Let $\vartheta := \theta^*$, and define a linear operator $\mathscr{T} : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathscr{T}(X) := \vartheta X \vartheta. \tag{4.2}$$

From (4.2) it is not obvious that the range of $\mathscr{T}$ is contained in $\mathfrak{g}$. To prove this, we recall some geometric constructions.

Let M,N be manifolds, let $\psi : M \to N$ be a diffeomorphism, and $X \in \mathrm{Vect}(M)$. Then

$$\psi^{-1*}X\psi^* = X(\cdot\circ\psi)\circ\psi^{-1}. \tag{4.3}$$

defines an operator on $C^\infty(N)$. One may check that this operator is a derivation, thus (4.3) defines a vector field on N. The vector field (4.3) is usually denoted

$$\psi_* X = d\psi(X_{\psi^{-1}(p)})$$

and referred to as the *push-forward* of X.

We now wish to show that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $\mathfrak{g}_\pm$ are the ± 1-eigenspaces of $\mathscr{T}$. This is proven by introducing an inner product on $\mathfrak{g}$ with respect to which $\mathscr{T}$ is self-adjoint. Let K be a nonempty compact subset of M. Endow $\mathfrak{g}$ with the inner product

$$(X,Y)_K = \int_K \langle X, Y\rangle \, dv, \tag{4.4}$$

where $\langle\, , \,\rangle$ is the metric on M and dv is the Riemannian volume measure. Since elements of $\mathfrak{g}$ are smooth vector fields, the function $\langle X, Y\rangle$ is smooth, hence bounded on any compact set K. Thus $(X,Y)_K$ is defined for all $X,Y \in \mathfrak{g}$.

Theorem 4.1 *Consider $\mathfrak{g}$ as a Hilbert space with inner product* (4.4)*. The operator $\mathscr{T} : \mathfrak{g} \to \mathfrak{g}$ is self-adjoint with $\mathscr{T}^2 = I$; hence*

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_- \tag{4.5}$$

as an orthogonal direct sum of Hilbert spaces, where $\mathfrak{g}_\pm$ are the ± 1-eigenspaces of $\mathscr{T}$. Further, $\partial_t \in \mathfrak{g}_-$ (hence $\dim(\mathfrak{g}_-) \geq 1$). Elements of $\mathfrak{g}_-$ have hermitian quantizations, while elements of $\mathfrak{g}_+$ have anti-hermitian quantizations.

Proof Write (4.2) as

$$\mathscr{T}(X) = \theta^{-1*} X \theta^* = \theta_* X \,. \tag{4.6}$$

Thus $\mathscr{T}$ is the operator of push-forward by θ. The push-forward of a Killing field by an isometry is another Killing field, hence the range of $\mathscr{T}$ is contained in $\mathfrak{g}$. Also, $\mathscr{T}$ must have a trivial kernel since $\mathscr{T}^2 = I$, and this implies that $\mathscr{T}$ is surjective. It follows from (4.6) that $\mathscr{T}$ is a Hermitian operator on $\mathfrak{g}$. Hence $\mathscr{T}$ is diagonalizable and has real eigenvalues which are square roots of 1. This establishes the decomposition (4.5). That elements of $\mathfrak{g}_-$ have hermitian quantizations, while elements of $\mathfrak{g}_+$ have anti-hermitian quantizations follows from Theorem 3.1.

One must not be tempted to speculate that $\mathfrak{g}_-$ consists only of ∂_t. In particular, $\dim(\mathfrak{g}_-) = 2$ for $M = \mathbb{H}_2$.

5 *G* is generated by reflection-invariant and reflected isometries

Let $G = \mathrm{Iso}(M)$ denote the isometry group of M, as above. Then G has a $\mathbb{Z}_2$ subgroup containing $\{1, \theta\}$. This subgroup acts on G by conjugation, which is just the action $\psi \to \psi^\theta := \theta\psi\theta$. Conjugation is an (inner) automorphism of the group, so

$$(\psi\phi)^\theta = \psi^\theta \phi^\theta, \qquad (\psi^\theta)^{-1} = (\psi^{-1})^\theta .$$

Definition 5.1 *We say that $\psi \in G$ is* **reflection-invariant** *if*

$$\psi^\theta = \psi,$$

and that ψ is **reflected** *if*

$$\psi^\theta = \psi^{-1}.$$

Let G_{RI} denote the subgroup of G consisting of reflection-invariant elements, and let G_R denote the subset of reflected elements.

Example 5.1 Let $z = x + it$ be a coordinate on $M = \mathbb{R}^2$; then time-reflection is complex conjugation, and rotations in the xt-plane are reflected isometries. Define $T_w z = z + w$. Since $\overline{T_w \overline{z}} = \overline{\overline{z} + w} = T_{\overline{w}} z$, it follows T_w is reflection-invariant if w is real, reflected if w is pure imaginary, and otherwise it is neither.

Note that G_{RI} is the stabilizer of the $\mathbb{Z}_2$ action, hence a subgroup. Also, G_R is closed under the taking of inverses and does contain the identity, but the product of two reflected isometries is no longer reflected unless they commute. Generally, the product of an element of G_R with an element of G_{RI} is neither an element of G_R nor of G_{RI}. Thus we have:

$$\{1, \theta\} \subset G_R \cup G_{RI} \subsetneq G.$$

Although it is not true that $G = G_R \cup G_{RI}$, it is true that the identity component of G is generated by $G_R \cup G_{RI}$.

Theorem 5.1 *Let G^0 denote the connected component of the identity in G. Then G^0 is generated by $G_R \cup G_{RI}$.*

Proof Since $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a direct sum of vector spaces (though not of Lie algebras), we have

$$G^0 = \big\langle \exp(\mathfrak{g}) \big\rangle = \big\langle \exp(\mathfrak{g}_+) \cup \exp(\mathfrak{g}_-) \big\rangle.$$

Choose bases $\{\xi_{\pm,i}\}_{i=1,\dots,n_\pm}$ for $\mathfrak{g}_\pm$ respectively. Then we have:

$$G^0 = \big\langle \{\exp(s\xi_{+,i}) : 1 \le i \le n_+,\, s \in \mathbb{R}\} \cup \{\exp(s\xi_{-,j}) : 1 \le j \le n_-,\, s \in \mathbb{R}\} \big\rangle.$$

Furthermore, $\exp(s\xi_{-,i})$ is reflected, while $\exp(s\xi_{+,i})$ is reflection-invariant, as we now prove. Fix $\xi_\pm \in \mathfrak{g}_\pm$, and note that

$$q_\pm(s) := \theta \exp(s\xi_\pm) \theta$$

is a one-parameter group of isometries, hence $q_\pm(s)$ corresponds to a unique element of $\mathfrak{g}$. This element is clearly $\vartheta \xi_\pm \vartheta$, and since $\xi_\pm \in \mathfrak{g}_\pm$, we infer that $\vartheta \xi_\pm \vartheta = \pm \xi_\pm$. Therefore

$$\exp(\pm s\xi_\pm) = q_\pm(s) \equiv \theta \exp(s\xi_\pm) \theta.$$

In particular, $U(s) = \exp(s\xi_-)$ satisfies the relation $\theta U(s) = U(s)^{-1}\theta$, i.e. $\exp(s\xi_-)$ is reflected. Similarly, $\exp(s\xi_+)$ is reflection-invariant.

Corollary 5.1 *The Lie algebra of the subgroup G_{RI} is $\mathfrak{g}_+$. In particular, $\mathfrak{g}_+$ is a Lie subalgebra of $\mathfrak{g}$.*

To summarize, the isometry group of a static space-time can always be generated by a collection of n $(= \dim \mathfrak{g})$ one-parameter subgroups, each of which consists either of reflected isometries, or reflection-invariant isometries. Each of these one-parameter subgroups is null-invariant

6 Construction of Unitary Representations

6.1 Self-adjointness of semigroups

In this section, we recall several results on self-adjointness of semigroups. Roughly speaking, these results imply that if a one-parameter family S_α of unbounded symmetric operators satisfies a semigroup condition of the form $S_\alpha S_\beta = S_{\alpha+\beta}$, then under suitable conditions one may conclude essential self-adjointness.

A theorem of this type appeared in a 1970 paper of Nussbaum [15], who assumed that the semigroup operators have a common dense domain. The result was rediscovered independently by Fröhlich, who applied it to quantum field theory in several important papers [5, 3]. For our intended application to quantum field theory, it turns out to be very convenient to drop the assumption that $\exists a$ such that the S_α all have a common dense domain for $|\alpha| < a$, in favor of the weaker assumption that $\bigcup_{\alpha>0} D(S_\alpha)$ is dense.

A generalization of Nussbaum's theorem which allows the domains of the semigroup operators to vary with the parameter, and which only requires the *union* of the domains to be dense, was later formulated and two independent proofs were given: one by Fröhlich [4], and another by Klein and Landau [13]. The latter also used this theorem in their construction of representations of the Euclidean group and the corresponding analytic continuation to the Lorentz group [14].

In order to keep the present article self-contained, we first define symmetric local semigroups and then recall the refined self-adjointness theorem of Fröhlich, and Klein and Landau.

Definition 6.1 *Let $\mathscr{H}$ be a Hilbert space, let $T > 0$ and for each $\alpha \in [0,T]$, let S_α be a symmetric linear operator on the domain $\mathscr{D}_\alpha \subset \mathscr{H}$, such that:*

(i) $\mathscr{D}_\alpha \supset \mathscr{D}_\beta$ *if* $\alpha \le \beta$ *and* $\mathscr{D} := \bigcup_{0<\alpha\le T} \mathscr{D}_\alpha$ *is dense in* $\mathscr{H}$,
(ii) $\alpha \to S_\alpha$ *is weakly continuous,*
(iii) $S_0 = I$, $S_\beta(\mathscr{D}_\alpha) \subset \mathscr{D}_{\alpha-\beta}$ *for* $0 \le \beta \le \alpha \le T$, *and*
(iv) $S_\alpha S_\beta = S_{\alpha+\beta}$ *on* $\mathscr{D}_{\alpha+\beta}$ *for* $\alpha, \beta, \alpha+\beta \in [0,T]$.

In this situation, we say that $(S_\alpha, \mathscr{D}_\alpha, T)$ is a symmetric local semigroup.

It is important that $\mathscr{D}_\alpha$ is *not* required to be dense in $\mathscr{H}$ for each α; the only density requirement is (i).

Theorem 6.1 ([13, 4]) *For each symmetric local semigroup $(S_\alpha, \mathscr{D}_\alpha, T)$, there exists a unique self-adjoint operator A such that*[2]

$$\mathscr{D}_\alpha \subset D(e^{-\alpha A}) \;\; \text{and} \;\; S_\alpha = e^{-\alpha A}|_{\mathscr{D}_\alpha} \;\; \text{for all } \alpha \in [0,T].$$

Also, $A \ge -c$ if and only if $\|S_\alpha f\| \le e^{c\alpha}\|f\|$ for all $f \in \mathscr{D}_\alpha$ and $0 < \alpha < T$.

[2] The authors of [4, 13] also showed that

$$\widehat{\mathscr{D}} := \bigcup_{0<\alpha\le S} \Big[\bigcup_{0<\beta<\alpha} S_\beta(\mathscr{D}_\alpha) \Big], \quad \text{where} \quad 0 < S \le T,$$

is a *core* for A, i.e. $(A, \widehat{\mathscr{D}})$ is essentially self-adjoint.

6.2 Reflection-Invariant Isometries

Lemma 6.1 *Let ψ be a reflection-invariant isometry and assume $\exists\, p \in \Omega_+$ such that $\psi(p) \in \Omega_+$. Then ψ preserves the positive-time subspace, i.e. $\psi(\Omega_+) \subseteq \Omega_+$.*

Proof Note that $\psi(\Sigma) \subseteq \Sigma$, for if not then choose $p \in \Sigma$ with $\psi(p) \notin \Sigma$. Without loss of generality, assume $\psi(p) \in \Omega_+$. Thus Ω_+ contains $(\theta\psi\theta)(p) = \theta\psi(p) \in \Omega_-$, a contradiction since $\Omega_- \cap \Omega_+ = \emptyset$. We used the fact that $\theta = \mathrm{id}$ on Σ. Hence ψ restricts to an isometry of Σ. It follows that the restriction of ψ to $M \setminus \Sigma$ is also an isometry. However, $M \setminus \Sigma = \Omega_- \sqcup \Omega_+$, where $\sqcup$ denotes the disjoint union. Therefore $\psi(\Omega_+)$ is wholly contained in either Ω_+ or Ω_-, as the alternative would violate continuity. The possibility that $\psi(\Omega_+) \subseteq \Omega_-$ is ruled out by our assumption.

Lemma 6.1 has the immediate consequence that if $\xi \in \mathfrak{g}_+$ then the one-parameter group associated to ξ is positive-time-invariant. This result plays a key role in the proof of Theorem 6.2.

6.3 Construction of Unitary Representations

The rest of this section is devoted to proving that the theory of symmetric local semigroups can be applied to the quantized operators on $\mathscr{H}$ corresponding to each of a set of 1-parameter subgroups of $G = \mathrm{Iso}(M)$. We proceed in two steps. The first step is to show that the 1-parameter subgroups of interest define operators on $\mathscr{H}$; for this we use Theorems 3.1, 3.2 and 6.1.

Theorem 6.2 *Let (M, g_{ab}) be a quantizable static space-time. Let ξ be a Killing field which lies in $\mathfrak{g}_+$ or $\mathfrak{g}_-$, with associated one-parameter group of isometries $\{\phi_\alpha\}_{\alpha\in\mathbb{R}}$. Then there exists a densely-defined self-adjoint operator A_ξ on $\mathscr{H}$ such that*

$$\widehat{\Gamma}(\phi_\alpha) = \begin{cases} e^{-\alpha A_\xi}, & \text{if } \xi \in \mathfrak{g}_- \\ e^{i\alpha A_\xi} & \text{if } \xi \in \mathfrak{g}_+. \end{cases}$$

Proof First suppose that $\xi \in \mathfrak{g}_-$, which implies that the isometries ϕ_α are reflected, and so $\Gamma(\phi_\alpha)^+ = \Gamma(\phi_\alpha)$. Define

$$\Omega_{\xi,\alpha} := \phi_\alpha^{-1}(\Omega_+).$$

For sufficiently small α, $\Omega_{\xi,\alpha}$ is a nonempty open subset of Ω_+, and moreover, as $\alpha \to 0^+$, $\Omega_{\xi,\alpha}$ increases to fill Ω_+ with $\Omega_{\xi,0} = \Omega_+$. These statements follow immediately from the fact that $\phi_\alpha(p)$ is continuous with respect to α, and ϕ_0 is the identity map.

Since $\phi_\alpha(\Omega_{\xi,\alpha}) \subseteq \Omega_+$, we infer that $\Gamma(\phi_\alpha)\mathscr{E}_{\Omega_{\xi,\alpha}} \subseteq \mathscr{E}_+$. By Theorem 3.2, $\Gamma(\phi_\alpha)$ has a quantization which is a symmetric operator on the domain

$$\mathscr{D}_{\xi,\alpha} := \Pi(\mathscr{E}_{\Omega_{\xi,\alpha}}).$$

Fix some positive constant a with $\Omega_{\xi,a}$ nonempty. Note that

$$\bigcup_{0<\alpha\le a} \Omega_{\xi,\alpha} = \Omega_+ \quad \Rightarrow \quad \bigcup_{0<\alpha\le a} \mathscr{E}_{\Omega_{\xi,\alpha}} = \mathscr{E}_+.$$

It follows that

$$\mathscr{D}_\xi := \bigcup_{0<\alpha\leq a} \mathscr{D}_{\xi,\alpha}$$

is dense in $\mathscr{H}$. This establishes condition (i) of Definition 6.1, and the other conditions are routine verifications. Theorem 6.1 implies existence of a self-adjoint generator A_ξ such that

$$\widehat{\Gamma}(\phi_\alpha) = \exp(-\alpha A_\xi) \text{ for all } \alpha \in [0,a].$$

This proves the theorem in case $\xi \in \mathfrak{g}_-$.

Now suppose that $\xi \in \mathfrak{g}_+$, implying that the isometries ϕ_α are reflection-invariant, and

$$\Gamma(\phi_\alpha)^+ = \Gamma(\phi_\alpha)^{-1} = \Gamma(\phi_{-\alpha}) \text{ on } \mathscr{E}.$$

Lemma 6.1 implies that $\Gamma(\phi_\alpha)\mathscr{E}_+ \subseteq \mathscr{E}_+$. By Theorem 3.1, $\Gamma(\phi_\alpha)$ has a quantization $\widehat{\Gamma}(\phi_\alpha)$ which is defined and satisfies

$$\widehat{\Gamma}(\phi_\alpha)^* = \widehat{\Gamma}(\phi_\alpha)^{-1}$$

on the domain $\Pi(\mathscr{E}_+)$, which is dense in $\mathscr{H}$ by definition. In this case we do not need Theorem 6.1; for each α, $\widehat{\Gamma}(\phi_\alpha)$ extends by continuity to a one-parameter unitary group defined on all of $\mathscr{H}$ (not only for a dense subspace). By Stone's theorem,

$$\widehat{\Gamma}(\phi_\alpha) = \exp(i\alpha A_\xi)$$

for A_ξ self-adjoint and for all $\alpha \in \mathbb{R}$. The proof is complete.

7 Analytic Continuation

Each Riemannian static space-time (M, g_{ab}) has a Lorentzian continuation M_{lor}, which we construct as follows. In adapted coordinates, the metric g_{ab} on M takes the form

$$ds^2 = \mathscr{F}(x)dt^2 + \mathscr{G}_{\mu\nu}(x)dx^\mu dx^\nu. \tag{7.1}$$

The analytic continuation $t \to -it$ of (7.1) is standard and gives a metric of Lorentz signature, ${ds^2}_{\text{lor}} = -\mathscr{F}\,dt^2 + \mathscr{G}\,dx^2$, by which we define the Lorentzian space-time M_{lor}. Einstein's equation $\text{Ric}_g = kg$ is preserved by the analytic continuation, but we do not use this fact anywhere in the present paper.

Let $\{\xi_i^{(\pm)} : 1 \leq i \leq n_\pm\}$ be bases of $\mathfrak{g}_\pm$, respectively. Let $A_i^{(\pm)} = A_{\xi_i^{(\pm)}}$ be the self-adjoint operators constructed by Theorem 6.2. Let

$$U_i^{(\pm)}(\alpha) = \exp(i\alpha A_i^{(\pm)}), \text{ for } 1 \leq i \leq n_\pm \tag{7.2}$$

be the associated one-parameter unitary groups.

We claim that the group generated by the $n = n_+ + n_-$ one-parameter unitary groups (7.2) is isomorphic to the identity component of

$$G_{\text{lor}} := \text{Iso}(M_{\text{lor}}),$$

the group of Lorentzian isometries. Since locally, the group structure is determined by its Lie algebra, it suffices to check that the generators satisfy the defining relations of $\mathfrak{g}_{\rm lor} := {\rm Lie}(G_{\rm lor})$.

Since quantization of operators preserves multiplication, we have

$$X,Y,Z \in \mathfrak{g},\ [X,Y] = Z \quad \Rightarrow \quad [\widehat{\Gamma}(X), \widehat{\Gamma}(Y)] = \widehat{\Gamma}(Z). \tag{7.3}$$

In what follows, we will use the notation $\widehat{\mathfrak{g}}_\pm$ for $\{\widehat{\Gamma}(X) : X \in \mathfrak{g}_\pm\}$, etc.

Quantization converts the elements of $\mathfrak{g}_-$ from skew operators into Hermitian operators; i.e. elements of $\widehat{\mathfrak{g}}_-$ are Hermitian on $\mathscr{H}$ and hence, elements of $i\widehat{\mathfrak{g}}_-$ are skew-symmetric on $\mathscr{H}$. Thus $\widehat{\mathfrak{g}}_+ \oplus i\widehat{\mathfrak{g}}_-$ is a Lie algebra represented by skew-symmetric operators on $\mathscr{H}$.

Theorem 7.1 *We have an isomorphism of Lie algebras:*

$$\mathfrak{g}_{lor} \;\cong\; \widehat{\mathfrak{g}}_+ \oplus i\widehat{\mathfrak{g}}_- \,. \tag{7.4}$$

Proof Let $M_\mathbb{C}$ be the manifold obtained by allowing the t coordinate to take values in $\mathbb{C}$. Define $\psi : M_\mathbb{C} \to M_\mathbb{C}$ by $t \mapsto -it$. Then $\mathfrak{g}_{\rm lor}$ is generated by

$$\{\xi_i^{(+)}\}_{1\le i\le n_+} \cup \{\eta_j\}_{1\le j\le n_-}, \quad \text{where} \quad \eta_j := i\psi^*\big(\xi_j^{(-)}\big).$$

It is possible to define a set of real structure constants f_{ijk} such that

$$[\xi_i^{(-)}, \xi_j^{(-)}] = \sum_{k=1}^{n_+} f_{ijk}\xi_k^{(+)}. \tag{7.5}$$

Applying ψ^* to both sides of (7.5), the commutation relations of $\mathfrak{g}_{\rm lor}$ are seen to be

$$[\eta_i, \eta_j] = -f_{ijk}\xi_k^{(+)}, \tag{7.6}$$

together with the same relations for $\mathfrak{g}_+$ as before. Now (7.3) implies that (7.6) are the precisely the commutation relations of $\widehat{\mathfrak{g}}_+ \oplus i\widehat{\mathfrak{g}}_-$, completing the proof of (7.4).

Corollary 7.1 *Let (M, g_{ab}) be a quantizable static space-time. The unitary groups* (7.2) *determine a unitary representation of ${G^0}_{lor}$ on $\mathscr{H}$.*

8 Hyperbolic Space and Anti-de Sitter Space

Consider Euclidean quantum field theory on $M = \mathbb{H}^d$. The metric is

$$ds^2 = r^{-2}\sum_{i=1}^{d} dx_i^2,$$

where we define $r = x^d$ for convenience. The Laplacian is

$$\Delta_{\mathbb{H}^d} = (2-d)r\frac{\partial}{\partial r} + r^2\Delta_{\mathbb{R}^d}\,. \tag{8.1}$$

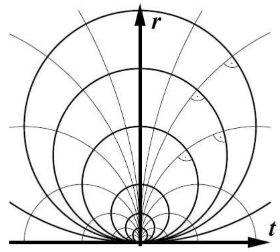

Fig. 1 Flow lines of the Killing field $\zeta = (t^2 - r^2)\partial_t + 2tr\,\partial_r$ on $\mathbb{H}^d$.

The $d-1$ coordinate vector fields $\{\partial/\partial x^i : i \neq d\}$ are all static Killing fields, and any one of the coordinates x^i $(i \neq d)$ is a satisfactory representation of time in this space-time. It is convenient to define $t = x^1$ as before, and to identify t with time.

The time-zero slice is $M_0 = \mathbb{H}^{d-1}$. From

$$\mathbb{H}^d = \{v \in \mathbb{R}^{d,1} \mid \langle v, v\rangle = -1, v_0 > 0\}$$

it follows that $\mathrm{Isom}(\mathbb{H}^d) = O^+(d,1)$ and the orientation-preserving isometry group is $SO^+(d,1)$.

For constant curvature spaces, one may solve Killing's equation $\mathscr{L}_K g = 0$ explicitly. Let us illustrate the solutions and their quantizations for $d = 2$. The three Killing fields

$$\xi = \partial_t, \quad \eta = t\partial_t + r\partial_r, \quad \zeta = (t^2 - r^2)\partial_t + 2tr\,\partial_r \tag{8.2}$$

are a convenient basis for $\mathfrak{g}$. Any d-dimensional manifold satisfies $\dim \mathfrak{g} \leq d(d+1)/2$, manifolds saturating the bound are said to be *maximally symmetric*, and $\mathbb{H}^d$ is maximally symmetric.

Now, $\partial_t f(-t) = -f'(-t)$ so $\partial_t \Theta = -\Theta \partial_t$, hence $\partial_t \in \mathfrak{g}_-$. Similar calculations show $[\Theta, \eta] = 0$ and $\Theta\zeta = -\zeta\Theta$. Thus η spans $\mathfrak{g}_+$, while ∂_t, ζ span $\mathfrak{g}_-$. The commutation relations[3] for $\mathfrak{g}$ are:

$$[\eta, \zeta] = \zeta, \quad [\eta, \partial_t] = -\partial_t, \quad [\zeta, \partial_t] = -2\eta.$$

The flows associated to (8.2) are easily visualized: ξ is a right-translation, and η flow-lines are radially outward from the Euclidean origin. The flows of ζ are Euclidean circles, indicated by the darker lines in Figure 1. It follows that the flows of η are defined on all of $\mathscr{E}_+$, while the flows of ζ are analogous to space-time rotations in $\mathbb{R}^2$, and hence, must be defined on a wedge of the form

$$W_\alpha = \{(t,r) \; : \; t, r > 0, \; \tan^{-1}(r/t) < \alpha\}.$$

The simple geometric idea of Section 6.2 is nicely confirmed in this case: the flows of η (the generator of $\mathfrak{g}_+$) preserve the $t = 0$ plane, and are separately isometries of Ω_+ and Ω_-.

Corollary 7.1 implies that the procedure outlined above defines a unitary representation of the identity component of $\mathrm{Iso}(AdS_2)$ on the quantum-field Hilbert space. In

[3] Note that quite generally $[\mathfrak{g}_-, \mathfrak{g}_-] \subseteq \mathfrak{g}_+$ so it's automatic that $[\zeta, \partial_t]$ is proportional to η.

general, $\mathrm{Iso}(AdS_{d+1}) = SO(d,2)$ so in this case, we have a unitary representation of $SO(1,2)_0$. The latter is a noncompact, semisimple real Lie group, and thus it has no finite-dimensional unitary representations, but a host of interesting infinite-dimensional ones.

A Euclidean Reeh-Schlieder Theorem

We prove the Euclidean Reeh-Schlieder property for free theories on curved backgrounds. It is reasonable to expect this property to extend to *interacting* theories on curved backgrounds, but it would have to be established for each such model since it depends explicitly on the two-point function.

The Reeh-Schlieder theorem guarantees the existence of a dense quantization domain based on any open subset of Ω_+. For this reason, one could use the Reeh-Schlieder (RS) theorem with Nussbaum's theorem [15] to construct a second proof of Theorem 6.2 under the additional assumption that M is real-analytic.

Fortunately, our proof of Theorem 6.2 is completely independent of the Reeh-Schlieder property. This has two advantages: we do not have to assume M is a real-analytic manifold and, more importantly, our proof of Theorem 6.2 generalizes immediately and transparently to interacting theories as long as the Hilbert space $\mathscr{H}$ is not modified by the interaction.

We state and prove this using the one-particle space; however, the result clearly extends to the quantum-field Hilbert space.

Theorem A.1 *Let M be a quantizable static space-time endowed with a real-analytic structure, and assume that g_{ab} is real-analytic. Let $\mathscr{O} \subset \Omega_+$ and $\mathscr{D} = C^\infty(\mathscr{O}) \subset L^2(\Omega_+)$. Then $\widehat{\mathscr{D}}^\perp = \{0\}$.*

Proof Let $f \in L^2(\Omega_+)$ with $\hat{f} \perp \mathscr{D}$. For $x \in \Omega_+$, define

$$\eta(x) := \langle \hat{f}, \hat{\delta}_x \rangle_{\mathscr{H}} = \langle \Theta f, C\delta_x \rangle_{L^2}.$$

Real-analyticity of $\eta(x)$ follows from the real-analyticity of (the integral kernel of) C, which in turn follows from the elliptic regularity theorem in the real-analytic category (see for instance [1, Sec. II.1.3]). Now by assumption, for any $g \in C_c^\infty(\mathscr{O})$, we have

$$0 = \langle \hat{g}, \hat{f} \rangle_{\mathscr{H}} = \langle \Theta f, Cg \rangle_{L^2(M)}.$$

Let $g \to \delta_x$ for $x \in \mathscr{O}$. Then $0 = \langle \Theta f, C\delta_x \rangle_{L^2} \equiv \eta(x)$. Since $\eta|_{\mathscr{O}} = 0$, by real-analyticity we infer the vanishing of η on Ω_+, completing the proof.

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