Holographic Software for Quantum Networks

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We introduce a new diagrammatic approach to quantum information, called *holographic software*. Our software captures both algebraic and topological aspects of quantum networks. It yields a bi-directional dictionary to translate between a topological approach and an algebraic approach. Using our software, we give a topological simulation for quantum networks. The string Fourier transform (SFT) is our basic tool to transform product states into states with maximal entanglement entropy. We obtain a diagrammatic interpretation of the phase space, of measurements, and of local transformations, including single-qudit Pauli matrices and their Jordan-Wigner transformations. We use our software to discover interesting new protocols for multipartite communication. In summary, we build a bridge linking the theory of planar para algebras with quantum information.

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I. INTRODUCTION

We introduce a new diagrammatic approach to quantum information, that we call *holographic software*. Our new approach is a small modification of previous diagrammatic approaches, yet our small change leads to a great deal of new understanding, and to a large number of new insights and motivation. Its application leads us to new protocols, designed in a "topological" way. Let us summarize two notions that we label P1 and P2:

(P1) The fundamental idea to use a diagrammatic notation to describe tensor manipulation originated in the work of Penrose [1]. This work has been extensively developed by Abramsky, Coecke, and their coworkers, yielding many diagrammatic representations of tensors and other algebraic structures in tensor networks [2, 3].

(P2) Many diagrams have topological meaning, and its importance in quantum information was recognized in the pioneering work of Kitaev, Freedman, Larsen, Wang, Kauffman, and Lomonaco [4–8]. People also investigated quantum computation [9–11]. In addition, the topological models of Kitaev, Levin, and Wen provide powerful tools in quantum information [4, 12].

Manin and Feynman introduced the concept of quantum simulation for quantum systems [13–15]. In our framework we simulate quantum networks using a topological model PAPPA constructed in [16]. This fits into the notion P2. In this paper we present PAPPA by what we call the *two-string model*. We anticipate treating a four-string PAPPA model on another occasion.

Our holographic software allows us to translate in both directions between the topological approach P2 and the algebraic approach P1. Thus we obtain new diagrammatic interpretations of algebraic structures, and vice-versa.

Recently the group around Pan achieved great success toward implementing long-distance quantum communication through the launch of "Micius," which plans the first test of quantum communication with a satellite [17]. Some interesting comments are in [18, 19]. These ideas for communication build upon fundamental work carried out on earth by the groups of Zeilinger and Pan [20–23], and Bennett et al. [24]. The quantum satellite "Micius" extends earlier work carried out in space by Vallone et al. [25], where they sent a photon from space to ground with photon polarization encoding, and Tang et al. [26] who measured the polarization correlation between pairs of photons in space and compared the results with similar experiments performed on earth.

Both subjects, quantum networks and quantum computation, are interesting in quantum information. In this paper we focus on the former. It would be interesting to simulate quantum algorithms [27] using our topological model.

A. Holographic Software

Let us explain what we mean by holographic software. Quantum information protocols are expressed as products of three types of elementary operations: unitary transformations on states, measurements, and classical communication of the results of measurements. These operations act on states, that are either inputs or given resource states. One can consider these operations and states as elementary network instructions.

Our diagrams are composed of charged strings. Our holographic software gives a dictionary to translate between diagrams and elementary network instructions. So we refer to these diagrams as elements of our software. Our software is universal in the sense that any algebraic protocol can be translated into diagrams. This approach is helpful to simplify algebraic computations and to notice the topological features of algebraic protocols. We are especially interested in combinations of these instructions that we can express with elementary diagrams.

Our present diagrammatic approach provides a way to pass in the inverse direction: from topology to algebra. This is the major new aspect of our work. We start from a given topological model, and use it to simulate processes in quantum information. In doing so, we can follow topological intuition to find new concepts and new protocols. We have resolved the technical difficulty of finding a robust and useful topological model for quantum information in [16].

For this reason, the diagrams that we treat here are conceptually different from the diagrams used previously in quantum information. They capture certain important features of protocols that are missing in earlier approaches. Our diagrams allow us to pass in both directions: algebra to topology, and topology to algebra. That is why we call it "holographic."

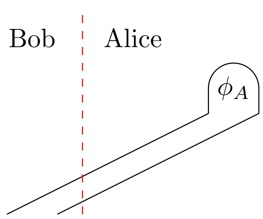
This philosophy also leads to new concepts and applications, such as the string Fourier transform and a pictorial representation of entanglement, Pauli matrices, and local transformations. One can introduce diagrammatic protocols and translate them into the usual algebraic protocols using our dictionary of the holographic software. This provides a new framework to explore the efficient simulation of physical systems.

B. Topological Simulation

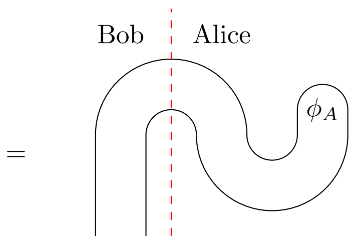
Let us illustrate this concept in the simulation of bipartite teleportation, our favorite example. We explain how we use our holographic software to recover in a natural way the resource state, measurement, and Pauli matrices—as well as the protocol of Bennett and coworkers [24]. If we simulate more complicated communication processes—by following topological intuition—then we find that our software leads to new concepts, as well as to new protocols in quantum information.

1. *Our Favorite Example*

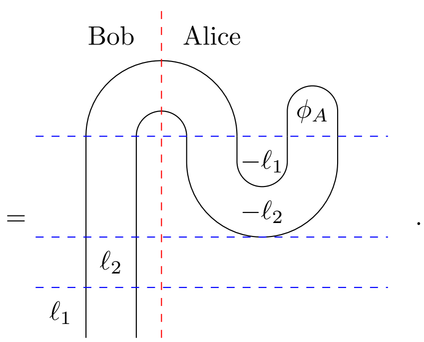
Suppose that Alice wants to teleport her qudit to Bob. That means she wants to use a protocol by which the quantum information, encoded in her qudit, is faithfully delivered to Bob's location. We outline the topological simulation of qudit teleportation here and explain the details in §III. Although every diagram has an interpretation in quantum information, it is important to recognize what diagrams have appropriate meanings in protocols.



(1)



(2)



(3)

We begin by drawing an elementary diagram that simulates the teleportation from Alice to Bob using a noiseless channel, as shown in (1). The dashed red line separates the communicating parties and is not a part of the diagram.

The disadvantage of the process (1) is that the noiseless channel is extremely expensive. And the information transmitted in this way may be intercepted by an adversary. Do we have a better simulation without the use of noiseless channels? Yes, the solution is to use topological isotopy, which does not change the function of the protocol, but it changes the way to implement it.

The solution to this problem is to simply make a topological isotopy that deforms the diagram into (2). What is the difference? Now the diagram extending over the red line is on the top of the picture. The double cap that extends over the dashed red line can be implemented by an entangled state shared by Alice and Bob.

This entangled state is a resource state in our protocol, which can be realized in quantum mechanics. One obtains a first estimate of the cost of the resource states in the protocol by simply counting one-half of the number of lines over the red dashed line.

Now the caps in (2) are a tensor product of two states. But what does the rest of the diagram represent? The double cup should be implemented by a measurement. In quantum information we cannot predict the outcome of the measurement. So to indicate the different possible outcomes we introduce the notion of *charge* on the strings.

The charges on the cups in (3) indicate the result of the measurements performed by Alice. One needs to make up the opposite charge on the strings on the left, so that the function of the diagram does not change. That means Alice needs to broadcast classically the results of her measurements to Bob, and Bob implements the corresponding recovery transformation to obtain a perfect replica at his site of the qudit that Alice teleports.

Now that we have obtained the diagrammatic protocol for qudit teleportation, let us translate this with our holographic software into the usual algebraic form. We will see how to recover from holographic software some fundamental concepts in quantum information:

Firstly, the entangled state expressed as the double cap is the standard resource state, namely the Bell state.

Secondly, the measurement expressed as the charged double cup is the measurement in the phase space. In addition, measurement arising in this pictorial way maps pure states to pure states.

Thirdly, the recovery transformations that arise on the left-hand strings are Pauli X, Y, Z matrices, given in (31).

We end up with the original algebraic teleportation protocol that was identified by Bennett et al [24], and we illustrate the protocol in Figure 1.

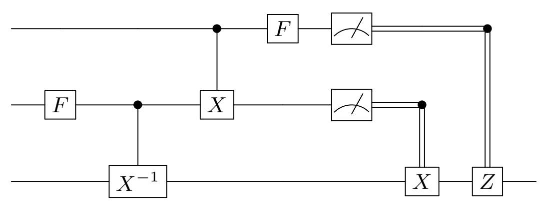


FIG. 1. Algebraic protocol for teleportation.

Therefore our topological simulation through holographic software is the reverse of the usual philosophy: algebra to topology. Instead of presupposing the notions of quantum information, we find that they arise naturally, from following the topological intuition in our model. After defining holographic software in the bulk of our paper, we give more details for this protocol in §IV A.

C. Two philosophies

The mathematical history of understanding the connection between algebra and topology goes back to the early 1900's in the development of homology theory. A quantum version of this philosophy arose thirty years ago. A breakthrough came when Vaughan Jones found quantum knot invariants [28–30]. In this theory, one gets diagrammatic representations for algebraic identities and topological invariants. This is the direction that we call: *algebra to topology.*

Jones then asked the question: can these invariants be derived in a topological manner, rather than in an algebraic way? In other words, can one go in the direction *topology to algebra*? Witten answered this by giving a topological interpretation to the Jones polynomial as an expectation of a Wilson loop in a field theory with a Chern-Simons action [31]. Actually Witten's picture is more general. It led to Atiyah's construction of a 3-d topological quantum field theory (TQFT) [32], which Reshetikhin, Turaev, and Viro formalized mathematically [33, 34]. The point of this story is that 3-d TQFT captures the algebraic axioms of modular tensor categories. As a concrete example, this includes representation categories of quantum groups.

In quantum information, one can imagine the same two philosophies (P1) and (P2). Our approach to communication captures both philosophies.

D. How is our approach new?

We have outlined above a number of features that are unique to our software. Now we attempt to give an overview of why our diagrammatic approach to quantum information is qualitatively different from prior studies with diagrams, and why we believe that it will have an important place in the future of the subject.

We have mentioned the extensive literature of diagrammatic work, based on the idea of using methods in the mathematical study of tensor categories. In their work they represent transformations on a qudit by strings (or rays) as inputs and outputs, composing them into algebraic structures. Our diagrams differ in several important ways.

- Our space of single qudits becomes a space of caps, rather than a space of points.
- Our single qubit transformations are represented by 2 strings rather than by a single string.
- Our strings carry charge.
- Our diagrams satisfy a para isotopy relation, rather than isotopy invariance.
- Our topological model comes from planar para algebras [16], rather than from tensor categories.
- The string Fourier transform (SFT) occurs naturally in our model, (but not in earlier frameworks).

We obtain a conceptual diagrammatic representation of many ideas, including: a picture of the string Fourier transform §III J, a picture of maximal entanglement, FIG. 2, a picture of the meaning of local transformations §III H 1, a picture of the Jordan-Wigner transformation §III H 2, new universal gate sets for quantum computation §V, etc. These and other phenomena that we can describe in pictures were not previously captured in such a natural way through a diagrammatic approach. The insights of our diagrams have given intuition that led us to generalize certain old protocols in a multipartite fashion §IV C, and to discover a new compressed teleportation protocol, see [37].

Furthermore we presented another model in [16] with four string transformations about which we plan to write more about in the future, and we hope this will result in new understanding of error correction. We also conjecture that these ideas will open up further understanding between the subjects of quantum information science and topological field theory.

E. String Fourier transform vs. the braid

Originally we had thought that the fundamental way to think about entanglement of qudits lay in the topological properties of the braid, for the braid allows consideration of isotopy in three dimensions. Many other authors have done great work using the braid in quantum information, and this is why we give so many references in that direction in §III G.

But after discovering holographic software, we have come to a different understanding. We now believe that the *string Fourier transform* (SFT) that we introduced in [16] provides a robust starting point for many aspects of quantum information, including entanglement. In addition, the SFT gives a new way to realize universal quantum computation and universal quantum simulation; see §III J and §V.

Our realization of the maximally-entangled, multipartite resource state, as well as our realization of maximal entanglement, is a consequence of the SFT. It comes from the SFT of the zero particle state. The algebraic formulas for the SFT and for the braid can be derived from one another. But we have learned to think about entanglement in terms of the SFT. And this provides insight into computations, and it yields simplification for a number of quantum information protocols; it also suggests new protocols.

Our SFT arose in the context of planar para algebras [16], before we understood the depth of its significance for quantum information. Geometrically, the SFT acts on diagrams and gives them a partial rotation. These diagrams might represent qudits, transformations, or measurements. The origin of SFT goes back to the work of Ocneanu in the more general context of work on subfactor theory [38].

1. *The maximally-entangled multipartite resource state*

In this paper we focus on a special subset of SFT's that transform n-qudits to n-qudits. Then the SFT acts as a very interesting unitary transformation $\mathfrak{F}_{\rm s}$ on the Hilbert space of n-qudits, that has dimension d^n. The transformation $\mathfrak{F}_{\rm s}$ applied to the n-qudit zero particle state $|\,\vec{0}\,\rangle$ creates the n-qudit $|\mathrm{Max}\rangle$. Briefly the standard n-qudit orthonormal basis $|\,\vec{k}\,\rangle$ is characterized by a set of charges $\vec{k}=(k_1,\ldots,k_n)$, with values $k_j\in\mathbb{Z}_d$, and with total charge $|\vec{k}|=k_1+\cdots+k_n$. In §III J we compute the matrix elements of $\mathfrak{F}_{\rm s}$ and show that

$$|\mathrm{Max}\rangle=\mathfrak{F}_{\rm s}\,|\,\vec{0}\,\rangle=\frac{1}{d^{\frac{n-1}{2}}}\sum_{|\vec{k}|=0}|\,\vec{k}\,\rangle\ . \tag{4}$$

In §III K we discuss definitions of entanglement entropy $\mathcal{E}$. A simple inequality shows that $|\mathrm{Max}\rangle$ maximizes this entropy, so $|\mathrm{Max}\rangle$ is a maximally-entangled n-qudit. The state $|\mathrm{Max}\rangle$ provides the natural multipartite analog of the Bell state.

We give the diagrammatic representation for $|\mathrm{Max}\rangle$ in Fig. 2. This is the multipartite[1] entangled resource state

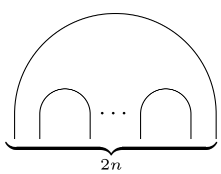


FIG. 2. Diagrammatic representation of the multipartite entangled state $|\mathrm{Max}\rangle$. There are $2n$ output points at the bottom.

for our protocols, which we discuss in §IV B, and which we use in our new protocol [37].

When the multipartite entangled state occurs in protocols, we indicate the corresponding n-qudit resource in Fig. 3.

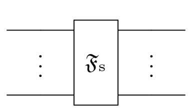


FIG. 3. Protocol for $|\mathrm{Max}\rangle$, the multipartite entangled resource state. There are n input and n output lines.

In the special case of order $d=2$ and $n=2$ (i.e. for 2-qubits), the matrix $\mathfrak{F}_{\rm s}$ plays the same role as the Hadamard transformation, followed by CNOT. This is the usual way to maximally entangle two qubits. In this case $|\mathrm{Max}\rangle$ is the Bell state.

F. The relation between $|\mathrm{Max}\rangle$ and $|\mathrm{GHZ}\rangle$

The state

$$|\mathrm{GHZ}\rangle=\frac{1}{d^{\frac{1}{2}}}\sum_{k=0}^{d-1}|k,k,\ldots,k\rangle \tag{5}$$

was considered as a multipartite resource state (originally for n-qubit entanglement) by Greenberger, Horne, and Zeilinger [20]. In §III L we show the entangled states $|\mathrm{GHZ}\rangle$ and $|\mathrm{Max}\rangle$ are related by a local transformation, i.e. by a simple tensor product of transformations. In fact $|\mathrm{GHZ}\rangle$ is the ordinary Fourier transform of $|\mathrm{Max}\rangle$,

$$|\mathrm{GHZ}\rangle=(F\otimes\cdots\otimes F)^{\pm1}|\mathrm{Max}\rangle\ , \tag{6}$$

with the Fourier transform F on a single qudit defined in (10).

G. Other key aspects of holographic software

Let us mention some other key aspects of holographic software that we explain in this paper. These features allow us to give a mosaic of diagrams that represent qudits, transformations, and measurements.

We have made very careful choices of our conventions. For instance we put the charge on the right side of a cap in (28); this corresponds to the choice of q rather than q^{-1} in (11). This also corresponds to the choice of decreasing basis in (36). We believe that it is difficult to change any of our choices, while preserving *all* the beautiful diagrammatic relations that we present here.

- We represent qudits, transformations, and measurements as diagrams with input points on the top and output points on the bottom.
- A 1-qudit is a cap; it has zero input points and 2 output points.
- We assign labels to the strings in our diagrams, representing "charge" on the string.
- *Para isotopy* generalizes topological isotopy and allows us to manipulate diagrams with charge.
- The diagram for a twisted product yields insight into para isotopy for charge-neutral subsystems.
- Braids can be defined in terms of planar diagrams and relate to entanglement.
- Charged diagrams can pass freely under our braids, but not over them.

[1] In general we use the term multipartite entanglement for entangled states on a Hilbert space composed of multiple subsystems (physics)/ subspaces (math). We use bipartite for two subsystems (Bell states), and tripartite for three subsystems.

- We obtain elementary diagrams for n-qudit Pauli matrices X, Y, Z.
- We represent local transformations diagrammatically.
- In the case that local transformations are Pauli matrices, one obtains an intuitive representation for the Jordan-Wigner transformation.

We refer persons interested in the mathematical theory behind our diagrams to the paper [16], in which we introduce the notion of "planar para algebras" and analyze them in detail. One also finds an explanation and motivation for the names we use for the diagrammatic relations, as well as proofs of these relations.

H. Other protocols

In (1) we illustrate our diagrammatic protocol for the standard teleportation. We introduce the multipartite entangled resource state $|\mathrm{Max}\rangle$. One can also construct the resource state by a generalized protocol of Bose, Vedral, and Knight [39] using minimal cost of edits, cdits, and time; see §IV B–§IV C. Our generalized BVK protocol is motivated by [39], as well as the problem of efficiently entangling nodes in a distributed quantum computer or a quantum internet [40].

In another paper [37], we give a new compressed teleportation (CT) protocol involving n-qudits. We discovered this protocol using holographic software. So we believe that the diagrams studied here provide an interesting paradigm for quantum information.

Pan and his coworkers have a massive space science project [17], and their group has already launched a Quantum Science Satellite, which produces bipartite resource states for long-distance communication [18]. We hope that the CT protocol can be used in multipartite communication in the future.

There are many other interesting protocols, for example [20, 41–56], and it would be nice to analyze such protocols using holographic software.

I. Does SFT provide quantum simulation?

In classical information theory, the Fourier transform F plays a central role, in particular in signal recovery; see [57] for a robust application. We propose that the SFT could play an analogous role in quantum information.

The question of quantifying the advantage of a quantum computer is a long-standing open problem [58, 59]. The landmark papers of Lloyd [60], Zalka [61], Abrams and Lloyd [62], and Somma et al. [63] provide foundational insights in the pursuit of quantum simulation. In [40, 58, 64, 65] one finds extensive references; experimental work on quantum simulation has been achieved [66–69].

II. BASIC ALGEBRAIC NOTATION

A. Qudits

A 1-qudit is a vector state in a d-dimensional Hilbert space, where d is the *degree* of the qudit. (The usual case of qubits corresponds to $d = 2$.) We denote an orthonormal basis using Dirac notation by $|k\rangle$. We call k the charge of the qudit, and generally $k \in \mathbb{Z}_d$, the cyclic group of order d.

The dual 1-qudit $\langle \ell|$ is a vector state in the dual space to the d-dimensional Hilbert space. And $\langle \ell|k\rangle = \delta_{\ell,k}$, where $\delta_{\ell,k}$ is the Kronecker delta.

The n-qudit space is the n-fold tensor product of the 1-qudit space. An orthonormal basis for n-qudits is $|\vec{k}\rangle = |k_1, k_2, \ldots, k_n\rangle$, where this ket has total charge $|\vec{k}| = k_1 + k_2 + \cdots + k_n$. The dual basis is $\langle \vec{\ell}|$. Every linear transformation on n-qudits can be written as a sum of the d^{2n} homogeneous transformations

$$M_{\vec{\ell},\vec{k}} = |\vec{\ell}\,\rangle\,\langle \vec{k}\,| \;, \quad \text{with charge} \quad |\vec{k}| - |\vec{\ell}| \;. \tag{7}$$

The matrix elements of $T = \sum_{\vec{k},\vec{\ell}} t_{\vec{\ell},\vec{k}}\, M_{\vec{\ell},\vec{k}}$ are just $t_{\vec{\ell},\vec{k}} = \langle \vec{\ell}|T|\vec{k}\,\rangle$.

B. The parafermion algebra

The *parafermion algebra* is a $*$-algebra with unitary generators c_j, which satisfy

$$c_j^d = 1 \quad \text{and} \quad c_j c_k = q\, c_k c_j \quad \text{for} \quad 1 \leqslant j < k \leqslant m. \tag{8}$$

Here $q \equiv e^{\frac{2\pi i}{d}}$, $i \equiv \sqrt{-1}$, and d is the order of the parafermion. Consequently $c_j^* = c_j^{-1} = c_j^{d-1}$, where $*$ denotes the adjoint. Majorana fermions arise for $d = 2$.

The Jordan-Wigner transformation is an isomorphism between the parafermion algebra with $2n$ generators and the n-fold tensor product of the $d \times d$ matrix algebra, the latter gives n-qudit transformations. Therefore, we can express n-qudit transformations as elements in the parafermion algebra.

C. Transformations of 1-qudits

Let $q^d = 1$ and $\zeta = q^{1/2}$ be a square root of q with the property $\zeta^{d^2} = 1$. Matrices X, Y, Z, F, G play an important role. Three of these are the qudit Pauli matrices

$$X|k\rangle = |k+1\rangle \;,\; Y|k\rangle = \zeta^{1-2k}|k-1\rangle \;,\; Z|k\rangle = q^k|k\rangle \;. \tag{9}$$

The Fourier matrix F and the Gaussian G are

$$F|k\rangle = \frac{1}{\sqrt{d}} \sum_{\ell=0}^{d-1} q^{k\ell}|\ell\rangle \;, \quad G|k\rangle = \zeta^{k^2}|k\rangle \;. \tag{10}$$

These matrices satisfy the relations

$$XY = qYX\ , \quad YZ = qZY\ , \quad ZX = qXZ\ , \tag{11}$$

$$XYZ = \zeta\ , \quad FXF^{-1} = Z\ , \quad GXG^{-1} = Y^{-1}\ . \tag{12}$$

D. Transformations of 2-qudits

1. *The multipartite entangled resource state*

We represent the multipartite entangled resource state for 2-qudits as

$$|\mathrm{Max}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k, -k\rangle\ .$$

We say it costs 1 edit if two persons use this entangled state in a protocol.

2. *Controlled gates*

We give the protocol for controlled transformations $C_{1,A}$ in Fig. 4 and $C_{A,1}$ in Fig. 5, for different control qudits.

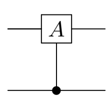


FIG. 4. The controlled gate $C_{1,A}$ acts on the 2-qudit $|k_1, k_2\rangle$ gives $C_{1,A}|k_1, k_2\rangle = \left|k_1, A^{k_1} k_2\right\rangle$. The first qudit is the control qudit.

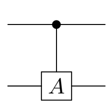


FIG. 5. The controlled gate $C_{A,1}$ acts on the 2-qudit $|k_1, k_2\rangle$ and gives $C_{A,1}|k_1, k_2\rangle = \left|A^{k_2} k_1, k_2\right\rangle$. The second qudit is the control qudit.

We sometimes allow more general controlled transformations of the form

$$T = \sum_{l=0}^{d-1} |\ell\rangle\langle\ell| \otimes T(\ell), \tag{13}$$

where the control is on the first qudit, and $T(\ell)$ can be arbitrary on the target qudit. This is shown in Fig. 6; a corresponding configuration with the second control bit would also possible.

The measurement controlled gate is illustrated in Fig. 7.

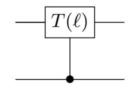


FIG. 6. Controlled transformations.

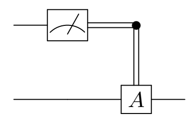


FIG. 7. Measurement controlled gate: If the qudit is measured by the meter as k, then they apply A^k to the target qudit. It costs 1 cdit to transmit the result, when the two qudits belong to different persons.

E. Qubit case: $d = 2$ and $\zeta = +i$

In the case $d = 2$ with $\zeta = +i = \sqrt{-1}$ the 1-qubit matrices X, Y, Z are the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$, while $F = H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the Hadamard matrix, and $G = S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ is the phase transformation. For 2-qubits, the transformation $C_{1,X}$ is CNOT. These transformations can be realized efficiently in nature [70–72]. The transformations they generate are local transformations.

F. Simplifying tricks

We give four elementary algebraic tricks to simplify the algebraic protocols; we illustrate them in Figs. 8–11.

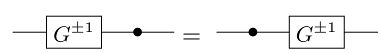


FIG. 8. Trick 1: The control gate commutes with the phase transformation on the control qudit.

III. HOLOGRAPHIC SOFTWARE

In this section we give the dictionary to translate between diagrammatic protocols and the algebraic ones. Any algebraic protocol can be translated into a diagrammatic protocol in a straightforward way. From this diagram we may be able to obtain new insights into the protocol.

We also give a dictionary for the inverse direction. Actually this is more interesting, as the diagrams may be more intuitive: one says that 1 picture is worth 1,000 words. In fact we give a new way to design protocols: we rely on the aesthetics of a diagram as motivation for the structure of the protocol. In this way, we can strive to introduce diagrammatic protocols which simulate human thought.

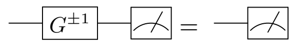


FIG. 9. Trick 2: The phase transformation does not affect measurement of the meter, so we can remove it.

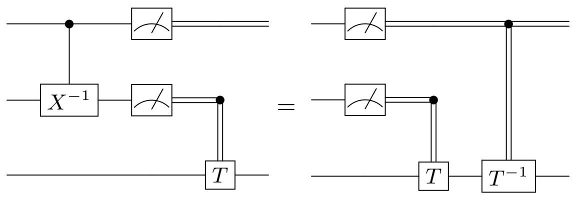


FIG. 10. Trick 3: We can remove the controlled transformation $C_{X,1}^{-1}$ before the double meters by changing the measurement controlled gate.

A. Diagrams for fundamental concepts

Before we give the complete list of diagrammatic relations and our dictionary for translation, let us remark how some fundamental concepts in quantum information fit into our diagrammatic framework.

In §II B we remark that one can write any n-qudit transformation as an element in the parafermion algebra with $2n$ generators. We represent the basis element $c_1^{k_1} c_2^{k_2} \cdots c_{2n}^{k_{2n}}$ in the parafermion algebra as a diagram with $2n$ "through" strings, with the j^{th} string labelled by k_j (on the left side). The label is called the charge of the string, and the labels are positioned in an increasing vertical order:

$$c_1^{k_1} c_2^{k_2} \cdots c_{2n}^{k_{2n}} = {}_{k_1}\Big| \; {}^{k_2}\Big| \cdots {}^{k_{2n}}\Big| \,. \tag{14}$$

The algebraic relations (8) to permute the order of factors in the product, become elementary relations between diagrams, that we will give in (17)-(18). Besides these relations mentioned here, we give other diagrammatic relations in §III C and §III G. In addition, we give examples of how to apply these relations to quantum information.

We can also represent n-qudits as diagrams. First we represent the n-qudit zero-particle state $|\vec{0}\,\rangle$, which up to a scalar is given by the n-cap diagram:

$$d^{n/4}\,|\vec{0}\rangle = \bigcap \quad \bigcap \quad \cdots \quad \bigcap \,. \tag{15}$$

The action of the parafermion algebra on the state $|\vec{0}\,\rangle$ is captured by the joint relations between the charged strings and the caps given in Equations (20).

It is extremely important that the multipartite entangled resource state $|\text{Max}\rangle$ (even in the case of multiple-persons) can be represented as the diagram in Fig. 2. This representation provides new insights in multipartite communication, which we explain later in this paper, and also in [37].

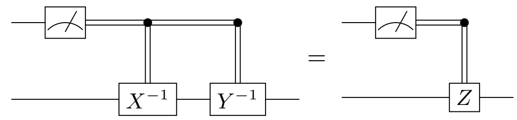


FIG. 11. Trick 4: Since $Y^{-i}X^{-i} = \zeta^{-i^2}Z^i$, and the phase does not count in the protocol, we can simplify meter-controlled transformations.

B. Elementary notions

We use a convention in identifying algebraic formulas with diagrammatic ones: the objects on the left side of an equation are represented by the objects on the right side of the equation.

In our diagrams, we call the points on top *input points*, and the points on bottom *output points*. The multiplication goes from bottom to top, and glues input points to output points. Tensor products go from left to right.

An n-qudit has 0 input points and $2n$ output points. A dual n-qudit has $2n$ input points and 0 output points.

We call a diagram with n input points and n output points an n-string transformation.

An n-qudit transformation is a $2n$-string transformation. (An n-qudit transformation is an n-string transformation in previous diagrammatic approaches, such as in [8, 36].) It is interesting that one can also talk about a 1-string transformation that acts on "$\frac{1}{2}$-qudits". We refer the readers to [37] for an application of this concept.

We call

$$k\Big| \tag{16}$$

a k-charged string, or a string with a k-charged particle. We write the label to the left of the string.

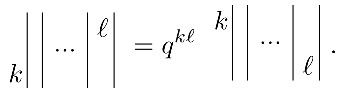
C. Planar relations

In this section we give relations between certain diagrams. The consistency of these relations is proved in [16]. Using these relations, we give a dictionary between qudits, transformations and diagrams.

1. Addition of charge, and charge order

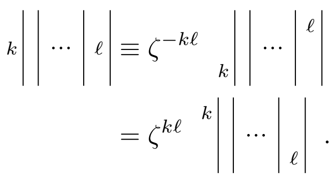
$$\begin{matrix} \ell\,\big| \\ k\,\big| \end{matrix} = k+\ell\,\Big| \,, \qquad d\,\Big| = \Big| \,. \tag{17}$$

2. Para isotopy

 (18)

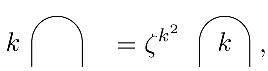
Here the strings between k^{th}-charged and ℓ^{th}-charged strings are not charged. We call $q^{k\ell}$ the twisting scalar.

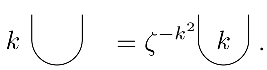
Notation: The twisted tensor product of pairs interpolates between the two vertical orders of the product. In the twisted product, we write the labels at the same vertical height:

 (19)

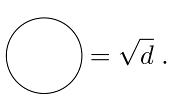
In this case $k, l \in \mathbb{Z}$, and k and $k+d$ yield different diagrams. If the pair is neutral, namely $\ell = -k$, then the twisted tensor product is defined for $k \in \mathbb{Z}_d$. This twisted product was introduced in [73, 74].

3. String Fourier relation

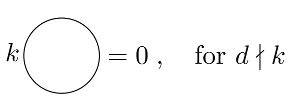
 (20)

 (21)

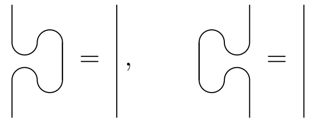
4. Quantum dimension

 (22)

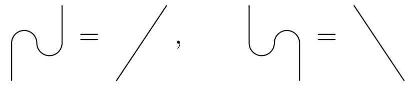
5. Neutrality

 (23)

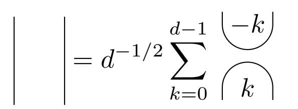
6. Temperley-Lieb relation

 (24)

Notation: Based on the Temperley-Lieb relation, a string only depends on the end points:

 (25)

7. Resolution of the identity

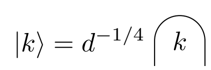
 (26)

D. 1-Qudit dictionary

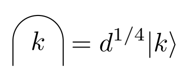
Now we give the first diagrammatic translations of the algebraic formulas. It will be evident from the context of the diagram, when a symbol such as k denotes a label, in contrast with $d^{-1/4}$ or $q^{k\ell}$ or ζ^{k^2}, that denote a scalar multiple.

1. Qudit

Our diagram for the qudit $|k\rangle$ is:

 (27)

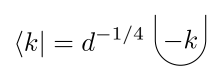
In other words, according to our convention,

 (28)

From now on, if the identification in both directions is clear, we only give one of them.

2. Dual qudit

Our diagram for the dual-qudit $\langle k|$ is:

 (29)

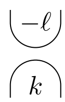
3. Transformations

Transformations T of 1-qudits are diagrams with two input points and two output points. The identity transformation is

$I =$  .

4. Matrix Units

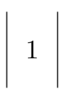
The diagram for the transformation $|k\rangle\langle\ell|$ is

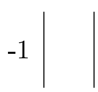
$|k\rangle\langle\ell| = d^{-1/2}$ 

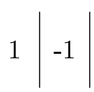
. (30)

5. Pauli matrices X, Y, Z

The diagrams for Pauli X, Y, Z are:

$X =$ 

, $Y =$ 

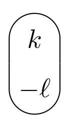
, $Z =$ 

. (31)

E. 1-Qudit properties

In this section we explain why the dictionary is holographic for 1-qudits, and we show how the Pauli X, Y, Z in (31) actually correspond to the usual qudit Pauli matrices.

- Orthonormal Basis:

  $\langle\ell|k\rangle = d^{-1/2}$ 

$= \delta_{\ell,k}$. (32)

  Here we use the relations (17), (22), and (23).

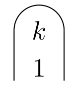
- Transformations: The matrix units $|k\rangle\langle\ell|$ are represented as in Equation 30.

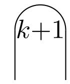
Therefore single qudit transformations can be represented as diagrams. On the other hand, Relation (26) indicates that any diagram with two input points and two output points is a single qudit transformation. This gives an elementary dictionary for translation between single qudit transformations and diagrams with two input points and two output points. In general, there is a correspondence between n-qudit transformations and diagrams with $2n$ input points and $2n$ output points.

In this way, the diagrammatic computation is the same as the usual algebraic computation in quantum information.

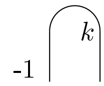
We introduce better diagrammatic representations for local transformations, so that we can utilize other diagrammatic relations.

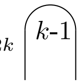
- Pauli X, Y, Z Relations: Using the notation for qudits in (28) and (31), one can identify these three 2-string transformations as the Pauli matrices defined in (9).

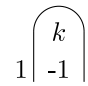
$=$ 

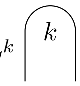
, (33)

$= \zeta^{1-2k}$ 

, (34)

$= q^k$ 

. (35)

  The diagrammatic equalities in (33)–(35) are a consequence of the relations (17)–(20).

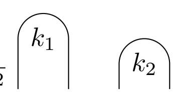
- Vertical reflection or Adjoint: The vertical reflection of diagrams maps the particle of charge k to the particle of charge $-k$. This involution is an anti-linear, anti-isomorphism of diagrams. It interchanges $|k\rangle$ with $\langle k|$. For qudits or transformations, the vertical reflection is the usual adjoint $*$.

F. n-Qudit dictionary

We mainly discuss the 2-qudit case. One can easily generalize the argument to the case of n-qudits.

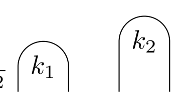
1. Elementary dictionary

There are two different ways to represent 2-qudits as diagrams indicated by the arrow.

$|k_1 k_2\rangle^{\searrow} = \frac{1}{d^{1/2}}$ 

, (36)

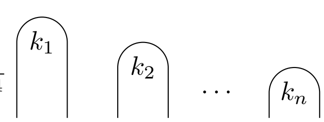
or

$|k_1 k_2\rangle^{\nearrow} = \frac{1}{d^{1/2}}$ 

. (37)

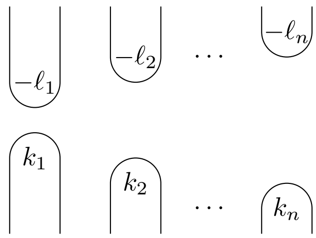
Then two representations give two different dictionaries, but they are unitary equivalent. We fix the first choice $|k_1 k_2\rangle = |k_1 k_2\rangle^{\searrow}$ in our software, since it works out better with concepts in quantum information.

We represent an n-qudit $|\vec{k}\rangle = |k_1, k_2, \cdots, k_n\rangle$ as

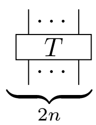
$|\vec{k}\rangle = \frac{1}{d^{n/4}}$ 

. (38)

We can represent the n-qudit transformation $|\vec{k}\rangle\langle\vec{\ell}| = |k_1, k_2, \cdots, k_n\rangle\langle\ell_1, \ell_2, \cdots, \ell_n|$ as

$$|\vec{k}\rangle\langle\vec{\ell}| = \frac{1}{d^{n/2}}$$  . (39)

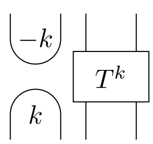
We denote an n-qudit transformation T as

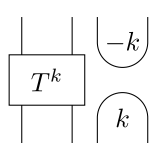
$T =$  . (40)

In the other direction, any diagram with 0 input points and $2n$-outpoint points is an n-qudit. Any diagram with $2n$ input points and $2n$ outpoint points is an n-qudit transformation.

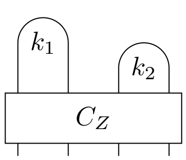
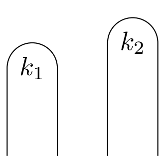
2. Controlled transformations

Suppose T is a single qudit transformation. Now we give the diagrammatic representation of the controlled transformations $C_{1,T}$ and $C_{T,1}$ in Figs. 4–5.

$$C_{1,T} = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}$$  , (41)

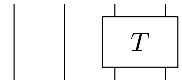
$$C_{T,1} = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}$$  . (42)

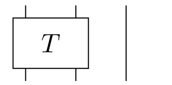
In particular, $C_Z \equiv C_{Z,1} = C_{1,Z}$, as one sees from its action on the basis $C_Z|k_1,k_2\rangle = q^{k_1k_2}|k_1,k_2\rangle$. Thus

 $=$  . (43)

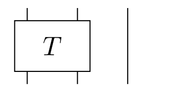
3. 1-Qudit transformations on 2-qudits

A 1-qudit transformation T can act on 2-qudits by adding two strings on the left or on the right. We can translate these diagrammatic transformations to algebraic ones as follows:

 $= 1 \otimes T$, (44)

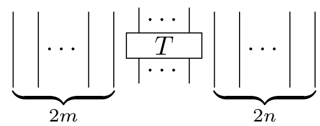
 $= C_Z(T\otimes 1)C_Z^{-1}$. (45)

Furthermore if T has charge k, then this action equals

 $= T\otimes Z^k$. (46)

Note (46) is better than (45), since Z^k and T can be performed locally by two persons.

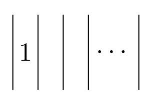
In general, if T has charge k, then

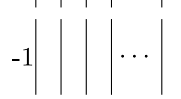
 (47)

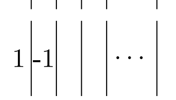
$$= \underbrace{1\otimes\cdots\otimes 1}_{2m} \otimes T \otimes \underbrace{Z^k\otimes\cdots\otimes Z^k}_{2n} . \tag{48}$$

4. Jordan-Wigner transformations

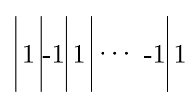
As a particular case of (47), we obtain the qudit Jordan-Wigner transformation for $T = X$, Y, or Z. We give an intuitive diagrammatic interpretation of this transformation in §III H.

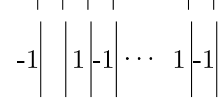
 $= X\otimes Z\otimes\cdots\otimes Z$, (49)

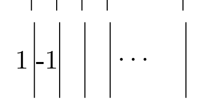
 $= Y\otimes Z^{-1}\otimes\cdots\otimes Z^{-1}$, (50)

 $= Z\otimes 1\otimes\cdots\otimes 1$. (51)

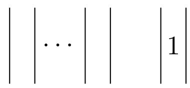
Equivalently, we can represent Pauli matrices on n-qudits as diagrams.

$X\otimes 1\otimes\cdots\otimes 1 =$  , (52)

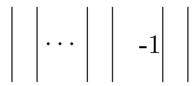
$Y\otimes 1\otimes\cdots\otimes 1 =$  , (53)

$Z\otimes 1\otimes\cdots\otimes 1 =$  . (54)

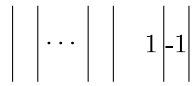
If we work on the increasing basis in Equation 37, then we obtain the following Jordan-Wigner transformation:



$$= Z^{-1} \otimes \cdots \otimes Z^{-1} \otimes X \,, \tag{55}$$



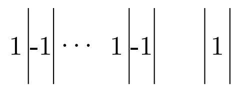
$$= Z \otimes \cdots \otimes Z \otimes Y \,, \tag{56}$$



$$= 1 \otimes \cdots \otimes 1 \otimes Z \,. \tag{57}$$

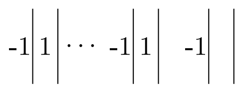
Equivalently,

$$1 \otimes \cdots \otimes 1 \otimes X = $$



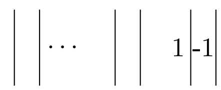
$$, \tag{58}$$

$$1 \otimes \cdots \otimes 1 \otimes Y = $$



$$, \tag{59}$$

$$1 \otimes \cdots \otimes 1 \otimes Z = $$

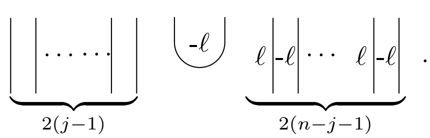


$$. \tag{60}$$

5. Measurement dictionary I

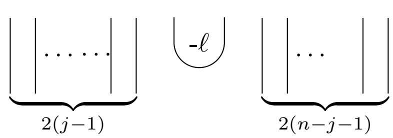
When a protocol has a meter, and the measurement of this meter is ℓ, it is the same as applying the dual qudit $\langle \ell |$ to the corresponding qudit. In a similar way, if the measurement of a meter m_j on the j^{th} qudit of an n-qudit is ℓ, then the diagram is

$$m_j = l \longrightarrow$$



$$. \tag{61}$$

Conversely, the diagram



$$\tag{62}$$

means that there is a meter on this j^{th} qudit of an n-qudit, and the measurement is ℓ. Moreover, the result is sent to persons who possess the last $(n-j-1)$ qudits. Then the persons apply $Z^{-\ell}$ to each of the $(n-j-1)$ target qudits. The corresponding protocol is in Fig. 12. Of course we can not predict the result of the measurement, so the diagrammatic protocol must work for all ℓ.

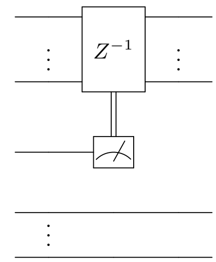


FIG. 12. Measurement controlled-Z gate: If the result of the measurement is k, then one applies Z^k to the target qudits.

G. Braided relations

1. Background

The topological approach to quantum computation became important with Kitaev's 1997 paper proposing an anyon computer—work that only appeared some five years later in print [4]. In §6 on the arXiv, he described the braiding and fusing of anyonic excitations in a fault-tolerant way. Freedman, Kitaev, Larsen, and Wang explored braiding further [5], motivated by the pioneering work of Jones, Atiyah, and Witten on knots and topological field theory [28–32].

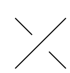
In the case $n = 2$, this braid appears in the Jones polynomial. For general n, these braids can be "Baxterized" in the sense of Jones [75]. They are the limits of solutions to the Yang-Baxter equation in statistical physics [76, 77], and have actually been introduced earlier by Fateev and Zamolodchikov [78]. Such kinds of braid statistics in field theory and quantum Hall systems were considered extensively by Fröhlich, see [79, 80]. Fermionic entanglement was addressed in [81, 82]. Kauffman and Lomonaco remarked that the braid diagram describes maximal entanglement [8]. From our point of view, it is natural to consider entanglement in terms of the string Fourier transform, see §III J.

2. The braid

We begin by defining a positive and negative braid in terms of planar diagrams. The braid acts on two strings. The justification for calling this diagram a braid, is that it satisfies the three Reidermeister moves characteristic of a braid. These relations allow one to lift the planar relations to three-dimensional ones. We refer the readers to [16] for the proof of the braided relations stated in this section.

Define $\omega = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \zeta^{j^2}$. Then ω is a phase, as shown in Proposition 2.15 of [16]. Let $\omega^{1/2}$ be a fixed square root of ω. Define the positive braid b_+ as

$$b_+ =$$

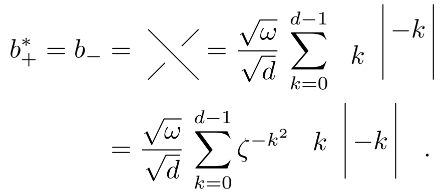


$$\equiv \frac{1}{\sqrt{\omega d}} \sum_{k=0}^{d-1} \; k \,\Big|\; {-k} \Big| \tag{63}$$

$$= \frac{1}{\sqrt{\omega d}} \sum_{k=0}^{d-1} \zeta^{k^2} \; k \,\Big|\; {-k} \Big| \;.$$

Here we give two different expressions for the braid. The second formula involves the twisted product given in (19).

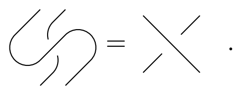
The braid is a unitary gate. Its adjoint b_+^* equals the inverse braid, the negative braid $b_+^{-1} = b_-$. In diagrams,



(64)

These definitions lead to the following braided relations:

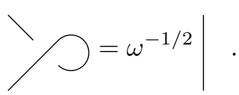
3. Braid-Fourier relation



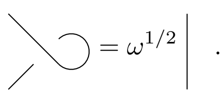
(65)

Thus drawing a braid at an arbitrary angle causes no confusion. This equation follows from (18), (21), (26), along with the identity $d^{-1/2}\sum_{k=0}^{d-1} q^{k\ell}\zeta^{k^2} = \omega\zeta^{-\ell^2}$.

4. Reidemeister move I

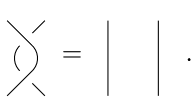


(66)



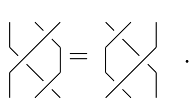
(67)

5. Reidemeister move II



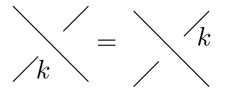
(68)

6. Reidemeister move III



(69)

7. The particle-braid relation

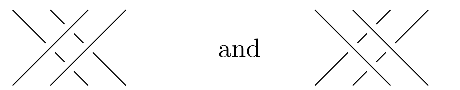


(70)

This relation demonstrates that any charged diagram can pass freely under (but not over) the braid.

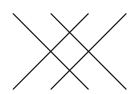
H. Two string braids and local transformations

From the 1-string braid constructed in §III G 2, we obtain a positive and a negative 2-string braid,



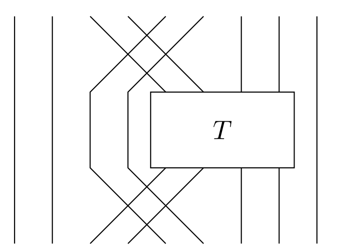
(71)

Theoretically, there are d different 2-string braids. Their actions on 2-qudits are defined by $b_m|k,l\rangle = q^{mkl}|l,k\rangle$, $m \in \mathbb{Z}_d$. Any neutral element can move over and under any two-string braid. Our positive braid is b_{-1} and our negative braid is b_1. Their interpolation b_0 is invariant under the 2-string rotation and adjoint operation, thus we represent this operator as



(72)

Since $b_0^2 = I$, we call it a symmetry. We use the symmetry b_0 to swap the order of the qudits. For example, if we want to apply a 2-qudit transformation T to the second and the fourth component of a 5-qudit, then we can represent the transformation diagrammatically as follows,



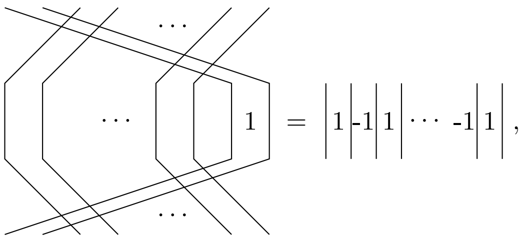
(73)

One can generalize this representation for arbitrary cases, meaning any number of qudits and any subsets.

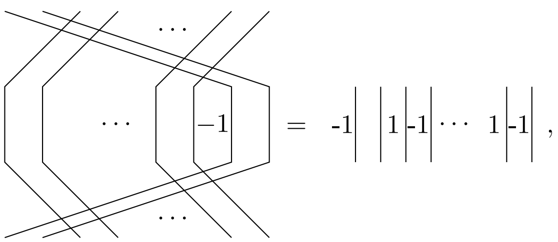
1. Local Transformations

If qudits in the subset belong to one person, then we call the transformation *local*. As examples of local transformations, we recover the Jordan-Wigner transformation for Pauli X, Y, Z in Equations (49),(50),(51) in terms of diagrammatic identities.

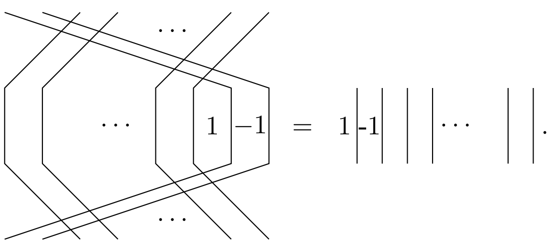
2. Jordan-Wigner as Local Transformations



(74)



(75)



(76)

I. SFT and maximal entanglement

In [16] we gave a general definition of the string Fourier transform $\mathfrak{F}_s$ on planar diagrams. Analytic properties of SFT have been studied in [83]. Here we analyze the special case of the SFT acting on n-qudits. In this case the transformation is given by a diagram with $2n$ input strings and $2n$ output strings, and it has charge 0. Acting on 2-qudits we illustrate $\mathfrak{F}_s$ in Fig. 13. The diagram for n-qudits is similar. We now analyze the SFT in more detail, both algebraically as well as with some relations for diagrams.

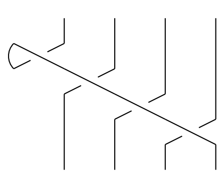
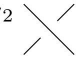
$\mathfrak{F}_s =$  (77)

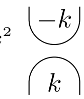
FIG. 13. String Fourier transform on 2-qudits.

1. String Fourier transform $\mathfrak{F}_s$ for 1-qudits

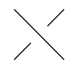
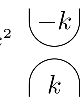
When $n = 1$, we infer from (20), (70), and (66), that

$\omega^{1/2}$  $= \mathfrak{F}_s = G.$ (78)

The positive and negative braids (63)–(64) also have the representations

 $= \frac{\sqrt{\omega}}{\sqrt{d}} \sum_{k=0}^{d-1} \zeta^{-k^2}$ 

, (79)

 $= \frac{1}{\sqrt{\omega d}} \sum_{k=0}^{d-1} \zeta^{k^2}$ 

. (80)

We conclude that $\mathfrak{F}_s$ and the braids act as local transformations on 1-qudits.

2. String Fourier transform $\mathfrak{F}_s$ on 2-qudits

In the $n = 2$ case, $\mathfrak{F}_s$ is a $d^2 \times d^2$ matrix. This matrix is block-diagonal, as it preserves the d different 2-qudit subspaces of fixed total charge, each of dimension d. We call $|0,0\rangle$ the zero particle state. The string Fourier transformation of the zero particle state is the maximally-entangled, multipartite, resource state

$$\mathfrak{F}_s|0,0\rangle = |\mathrm{Max}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k, -k\rangle \ . \tag{81}$$

The diagrammatic representation of the resource state is in Fig. 14. We recover the algebraic form of the resource state from its topological feature.

We use $|\mathrm{Max}\rangle$ as a resource to connect diagrams belonging to two persons in a quantum network. In a communication protocol between Alice and Bob, only strings of the resource state are allowed to connect them. Using the resource state costs 1 edit.

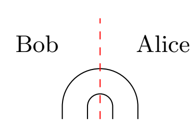


FIG. 14. Diagrammatic resource state: Only the strings in the resource state are allowed to pass the red (dashed) line between Alice and Bob. (The red line is only for explanation, not a part of the protocol.)

On 2-qudits, $\mathfrak{F}_s$ is a local transformation,

$$\begin{aligned} \mathfrak{F}_s &= (G^{-1} \otimes G) C_{1,X}^{-1} (F \otimes 1) C_{1,X} \ , \\ &= C_{X,1}^{-1} (1 \otimes F) C_{X,1} (G \otimes G^{-1}) \ . \end{aligned} \tag{82}$$

Note that $G^{-1} \otimes G$ is identity on 0-charge 2-qudits, so

$$\mathfrak{F}_{\mathrm{s}}|0,0\rangle = C_{1,X}^{-1}(F \otimes 1)|0,0\rangle \ . \tag{83}$$

The right side of this expresstion is the original formula for the resource state.

We have shown that the negative braid

acts on a qudit basis $|\ell\rangle$ as a local transformation $\omega^{-1/2}G$. It acts on the second and third strings of a 2-qudit as

$$b_{2,3,-} = \tag{84}$$

Then

$$\begin{aligned} b_{2,3,-} &= \omega(1 \otimes G^{-1})\mathfrak{F}_{\mathrm{s}}(G^{-1} \otimes 1) \\ &= \omega C_{1,X}^{-1}(G^{-1}FG^{-1} \otimes 1)C_{1,X} \\ &= \omega C_{X,1}^{-1}(1 \otimes G^{-1}FG^{-1})C_{X,1} \ . \end{aligned}$$

Thus $b_{2,3,-}$ is a local transformation.

J. String Fourier transform $\mathfrak{F}_{\mathrm{s}}$ for general n-qudits

In this section we show that the SFT acts on n-qudits as a unitary transformation $\mathfrak{F}_{\mathrm{s}}$. We find the matrix elements $\langle \vec{\ell}\,|\,\mathfrak{F}_{\mathrm{s}}|\vec{k}\rangle$ of $\mathfrak{F}_{\mathrm{s}}$ in (87)–(89). We also give the matrix elements of its inverse $\mathfrak{F}_{\mathrm{s}}^{*}$ in (90). Furthermore, we use these matrix elements to establish the fundamental relation in (92), namely that the SFT produces the maximally-entangled state from the zero-particle qudit,

$$|\mathrm{Max}\rangle = \mathfrak{F}_{\mathrm{s}}|\vec{0}\rangle = \mathfrak{F}_{\mathrm{s}}^{*}|\vec{0}\rangle = \frac{1}{d^{\frac{n-1}{2}}} \sum_{|\vec{k}|=0} |\vec{k}\,\rangle \ . \tag{85}$$

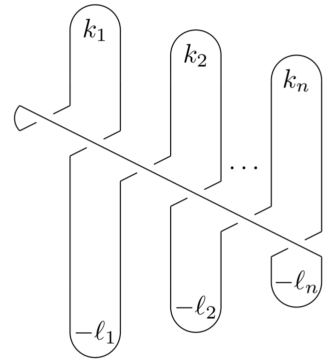
In addition, we obtain some other interesting properties of the SFT operator.

The diagram in Fig. 13 suggests that there is another formula for $\mathfrak{F}_{\mathrm{s}}$ given by the braid. Let $b_{i,i+1,-}$ be the negative braid on the i^{th} and $(i+1)^{\mathrm{th}}$ string. Each such transformation is local. Therefore we obtain the representation of the string Fourier transformation as the local transformation on n qudits,

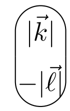
$$\mathfrak{F}_{\mathrm{s}} = \frac{1}{\sqrt{\omega}} b_{2n-1,2n,-}\, b_{2n-2,2n-1,-} \cdots b_{1,2,-} \ , \tag{86}$$

with the order in the product for increasing indices from right to left.

We calculate the matrix elements $\langle \vec{\ell}\,|\,\mathfrak{F}_{\mathrm{s}}|\vec{k}\rangle$ of $\mathfrak{F}_{\mathrm{s}}$ in the qudit basis $|\vec{k}\rangle = |k_1, k_2, \cdots, k_n\rangle$, and the dual qudit basis $\langle\vec{\ell}| = \langle \ell_1, \ell_2, \cdots, \ell_n|$. The diagrammatic answer is given in (87), namely



$$= \omega_{\vec{\ell},\vec{k}}$$



$$, \tag{87}$$

where

$$\omega_{\vec{\ell},\vec{k}} = \zeta^{|\vec{\ell}|^2} \prod_{1\leq j_1<j_2\leq n} q^{-\ell_{j_1}k_{j_2}} \ . \tag{88}$$

Thus the transformation $\mathfrak{F}_{\mathrm{s}}$ can be realized as a $d^n \times d^n$ matrix, with matrix elements

$$\langle \vec{\ell}\,|\,\mathfrak{F}_{\mathrm{s}}|\vec{k}\rangle = d^{\frac{1-n}{2}}\, \omega_{\vec{\ell},\vec{k}}\ \delta_{|\vec{\ell}|,|\vec{k}|} \ . \tag{89}$$

Similarly the matrix elements of the inverse string Fourier transformation on n-qudits are

$$\langle \vec{\ell}|\mathfrak{F}_{\mathrm{s}}^{-1}|\vec{k}\rangle = d^{\frac{1-n}{2}}\, \overline{\omega}_{\vec{\ell},\vec{k}}\ \delta_{|\vec{\ell}|,|\vec{k}|} \ . \tag{90}$$

Moreover,

$$\mathfrak{F}_{\mathrm{s}}^{2n}\,|\vec{k}\,\rangle = q^{|k|^2}\,|\vec{k}\rangle \ . \tag{91}$$

The string Fourier transform and its inverse map n-qudit product states to maximally entangled states. In particular, if $\vec{0} = (0, 0, \ldots, 0)$, we call $|\vec{0}\,\rangle$ the "zero particle state." Apply $\mathfrak{F}_{\mathrm{s}}$ to this state, and insert (89) with $\vec{k} = \vec{0}$, to obtain the multipartite resource state,

$$|\mathrm{Max}\rangle = \mathfrak{F}_{\mathrm{s}}|\vec{0}\,\rangle = \sum_{\vec{\ell}} \langle\vec{\ell}|\mathfrak{F}_{\mathrm{s}}|\vec{0}\rangle\,|\vec{\ell}\rangle = \frac{1}{d^{\frac{n-1}{2}}} \sum_{|\vec{k}|=0} |\vec{k}\,\rangle \ . \tag{92}$$

The coefficients of $|\vec{k}\,\rangle$ in the sum in (92) are all positive, because we have chosen the decreasing basis $|\vec{k}\rangle^{\searrow}$ for our qudits. Similarly $|\mathrm{Max}\rangle = \mathfrak{F}_{\mathrm{s}}^{*}|\vec{0}\,\rangle$.

We say that a protocol costs 1 n-edit, when it uses this n-qudit $|\mathrm{Max}\rangle$ as a resource. The diagrammatic representation and the protocol for the resource state are given in Fig. 2 and 3.

K. Entropy for n-qudit entanglement

There are several possible ways to define the entanglement entropy for multi-qudits. We give one particular definition for an n-qudit density matrix ρ. Let S denote an element of $\{1, 2, \ldots, n\}$ and S' its complement. Define the entanglement entropy for the set S as

$$\mathcal{E}_S(\rho) \equiv \mathcal{E}(\mathrm{tr}_{S'}(\rho)) \ , \tag{93}$$

where $\mathcal{E}$ denotes the von Neumann entropy and $\mathrm{tr}_{S'}$ denotes the partial trace on S'. This generalizes the definition in the 2-qudit case.

Then

$$\mathcal{E}_S(\rho_{\mathrm{Max}}) = -\frac{1}{d^{|S|}} \ln \frac{1}{d^{|S|}} \ , \tag{94}$$

where ρ_{Max} is the density matrix corresponding to the state $|\mathrm{Max}\rangle$. Elementary calculus shows that the entropy achieves its maximum among the set of pure states at $|\mathrm{Max}\rangle$, so this is the origin of the name.

L. The resource states $|\mathrm{Max}\rangle$ and $|\mathrm{GHZ}\rangle$

Here we establish the relation stated in (6), between the maximally-entangled resource state $|\mathrm{Max}\rangle$ and $|\mathrm{GHZ}\rangle$. In particular, these states are the ordinary Fourier transform $F \otimes \cdots \otimes F$ of one-another. To see this, use the representation (92) for $|\mathrm{Max}\rangle$. Then

$$(F \otimes \cdots \otimes F)|\mathrm{Max}\rangle = \frac{1}{d^{\frac{n-1}{2}+\frac{n}{2}}} \sum_{\substack{\vec{\ell} \\ \vec{k}: k_n = -k_1 - \cdots - k_{n-1}}} q^{\vec{k}\cdot\vec{\ell}} |\vec{\ell}\rangle \ .$$

Carry out the $(n-1)$ sums over $k_1, \ldots, k_{n-1}$ for fixed $\vec{\ell}$. These sums vanish unless $\ell_j = \ell_n$, for each $j = 1, \ldots, n-1$. In case all the equalities hold, there are d^{n-1} equal, non-zero terms. Thus the answer is as claimed in (6), namely

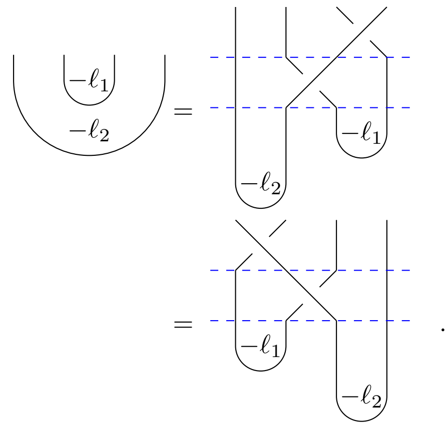
$$(F \otimes \cdots \otimes F)|\mathrm{Max}\rangle = \frac{1}{d^{\frac{1}{2}}} \sum_{\ell} |\ell, \ldots, \ell\rangle = |\mathrm{GHZ}\rangle \ . \tag{95}$$

The same result would arise with F^{-1} in place of F.

Up to a unitary equivalence, $|\mathrm{Max}\rangle$ and $|\mathrm{GHZ}\rangle$ are the same state. But we prefer the state $|\mathrm{Max}\rangle$ as the resource state for n-qudits, rather than $|\mathrm{GHZ}\rangle$, because $|\mathrm{Max}\rangle$ both has a *topological interpretation*, and in addition it is *neutral.*

M. Measurement dictionary II

We give a dual 2-qudit as a double-cup diagram in (96), (97). The two corresponding protocols are given in Fig. 15, 16 depending on the choice of the control qudit. They are equivalent to the protocol for measurement in phase space. This measurement is the most common measurement in protocols, known as the Bell state measurement for the qubit case. Thus we recover the measurement from its topological structure. This is the reverse of the historical route to go from the algebraic measurement to its topology.



(96)

(97)

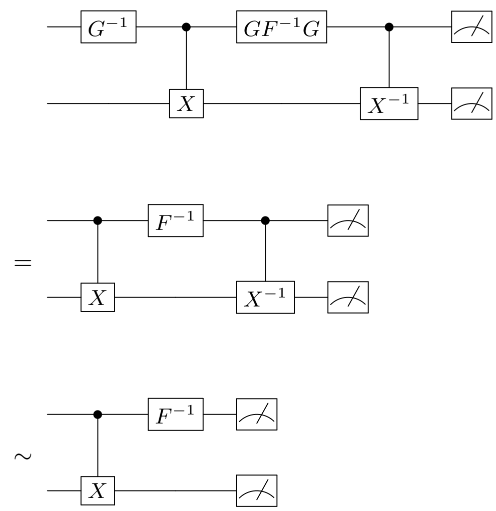


FIG. 15. Measurement in the phase space: The first protocol is translated from the double-cup diagram on the right of (96), where the measurement of the first and the second meters are ℓ_1 and ℓ_2 respectively. It is simplified as the second protocol using tricks in Figs. 8, 9. It is equivalent to the measurement in the phase space using the trick in Fig. 10.

IV. DIAGRAMMATIC IDENTIFICATION FOR PROTOCOLS

Now we complete the dictionary of our holographic software. We can use this dictionary to translate diagrammatic protocols to algebraic ones.

In this section we illustrate the robustness of the diagrammatic method, by giving examples. We identify the standard teleportation protocol. As mentioned in the introduction, in a separate paper we present the new compressed teleportation (CT) protocol.

Here we also construct a protocol to produce the multipartite entangled resource state $|\mathrm{Max}\rangle$ for n persons. This protocol requires using $(n-1)$ usual 2-edits, and

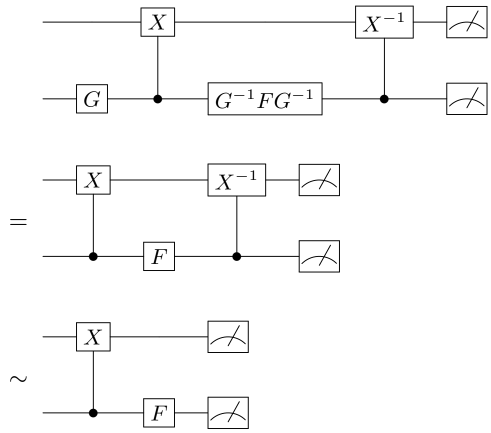
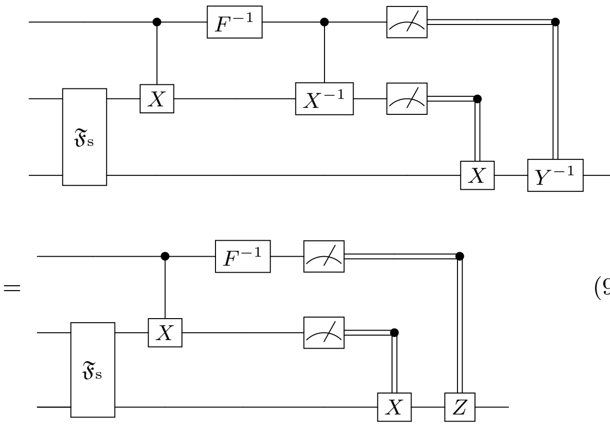


FIG. 16. Measurement in the phase space. This protocol is a translation of (97).



(99)

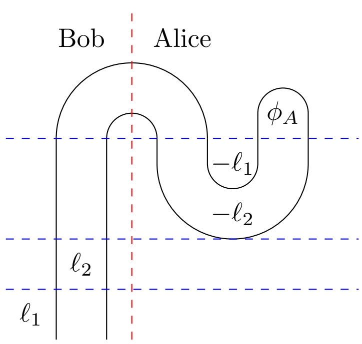
FIG. 17. Teleportation protocol: Measurement in the phase space: The first protocol represents the holographic translation of the diagrammatic protocol (98). It can be simplified to the protocol (99) using tricks in Figs. 10 and 11.

$(n-1)$ cdits. This cost is minimal, as is the cost in time, which is the transmission of one cdit.

When we translate between a diagrammatic realization of a protocol and an algebraic realization of that protocol, an overall (global) phase is irrelevant. It does not affect a quantum-mechanical vector state, even though in this paper we often do keep track of this phase.

A. Teleportation

As mentioned in the introduction, the diagram for standard qudit teleportation is



. (98)

Using our dictionary, we can translate the diagrammatic protocol in (98) piecewise to an algebraic protocol illustrated in Fig. 17. When $d = 2$, it is exactly the original qubit teleportation protocol of Bennett et al [24].

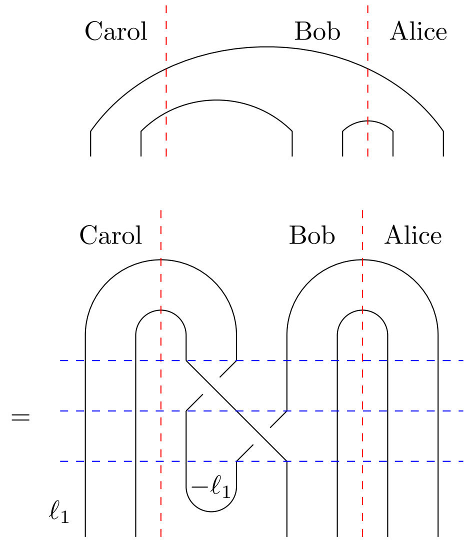
B. Multipartite resource state

We introduce the multipartite entangled resource state in (92). We can construct this n-qudit resource state using $(n-1)$ of the 2-qudit resource states. We give the diagrammatic protocol in (100) and the algebraic protocol in Fig. 18 for the case $n = 3$. One can easily generalize the protocol to the case for arbitrary n.

For the case $n = 3$ this entanglement protocol indicates how to construct a corresponding swapping protocol. It also shows that the usual swapping protocol wastes entanglement.

The point is that the usual swapping protocol uses the resource state between Alice and Bob, as well as the resource state between Bob and Carol. The result is a resource state between Alice and Carol. However, using our protocol we construct one resource state among the three persons: Alice, Bob, and Carol. In this way we do not lose the entanglement between Alice and Bob or between Bob and Carol. We can recover the resource state between one pair by measuring the qudit of the third person.

Our protocol for constructing the multipartite entangled resource state costs minimal edits. However, it is better to construct the multipartite entangled resource state as quantum software [44, 84] at a station and teleport each component to one person by a noiseless channel. This uses n noiseless channels in total. On the other hand, the construction of $(n-1)$, 2-qudit resource states uses $2(n-1)$ noiseless channels. Therefore, one may save cost by using n-qudit resource states for multipartite communication. Actually, it does save 50% in our new protocol given in [37].



$$. \qquad (100)$$

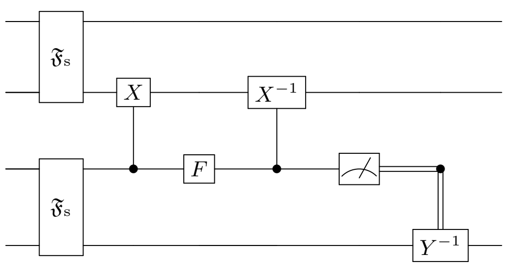
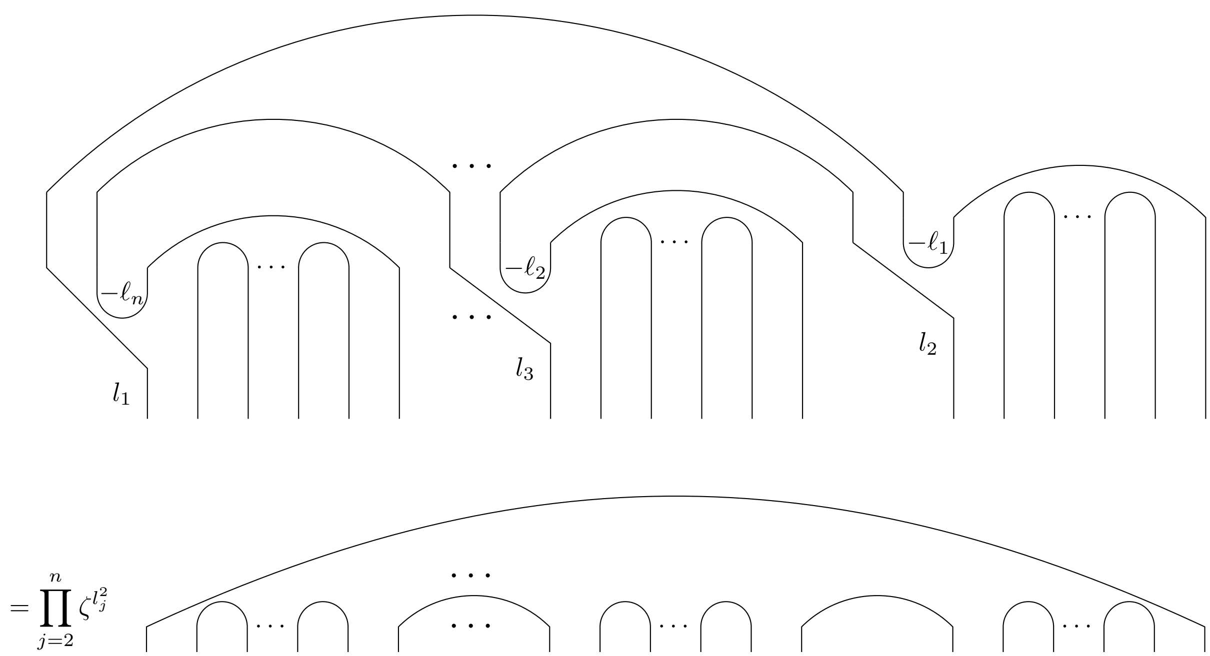


FIG. 18. The construction of the n-edit resource for $n = 3$.

C. The BVK protocol

Here we give a more general construction of $|\mathrm{Max}\rangle$ for a multipartite network, motivated by the Bose-Vedral-Knight protocol [39], and the challenge of Kimble to entangle nodes across a network for a quantum internet [40]. Suppose there are n parties and the j^{th} party has n_j persons with a shared multipartite entangled resource state $|\mathrm{Max}\rangle$. In each party there is one leader who shares an extra multipartite entangled resource state $|\mathrm{Max}\rangle$. Then we can construct a multipartite entangled resource state $|\mathrm{Max}\rangle$ for all members among the n parties. We illustrate this situation with a diagrammatic protocol in (101). We illustrate the corresponding algebraic protocol in Fig. 19.



$$(101)$$

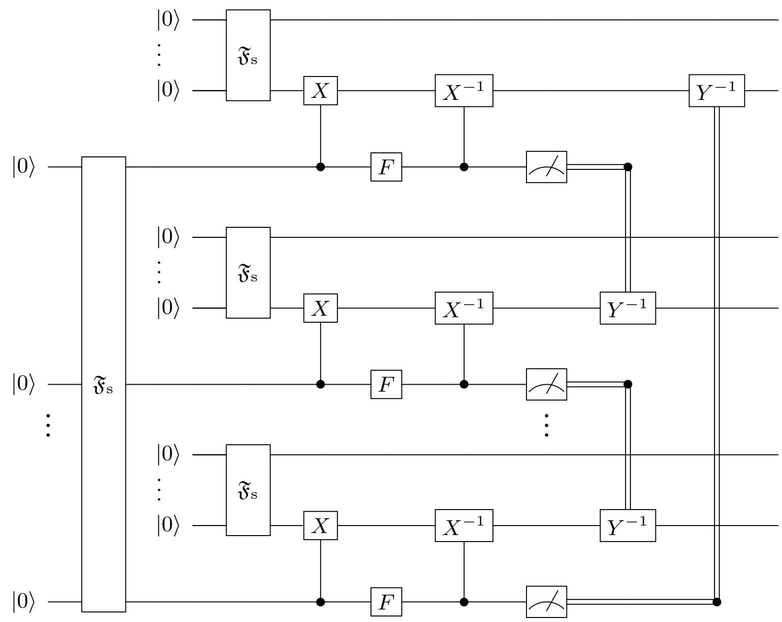


FIG. 19. The algebraic protocol for the iterated construction of the multipartite entangled resource state for multipartite communication corresponding to the diagram in (101).

V. THE PAPPA MODEL OF QUANTUM COMPUTATION

As an outgrowth of our investigation of quantum networks, we can comment on quantum computation. We show that the set of transformations generated by the one-qudit transformations X, Y, Z, F, G, along with the string Fourier transform $\mathfrak{F}_{\mathrm{s}}$ on 2-qudits is the Clifford group. The string Fourier transform on 2-qudits gives the controlled transformation C_Z as

$$C_Z = (GF^{-1} \otimes FG^{-1})\mathfrak{F}_{\mathrm{s}}(1 \otimes F^{-1}G^{-1}). \qquad (102)$$

It is known that the one-qudit transformations X, Y, Z, F, G with C_Z (or equivalently with C_X) generate the n-qudit Clifford group; see Theorem 7 of [85]. Thus, $X, Y, Z, F, G, \mathfrak{F}_{\mathrm{s}}$ generate the n-qudit Clifford group, and we refer to the matrices in this group as Clifford gates.

The n-qudit Clifford group is known to be insufficient for universal quantum computation, since the set of Clifford gates is not dense in the special unitary group $SU(d^n)$ of degree d^n. Moreover, the Gottesman-Knill theorem states that any quantum circuit built from stabilizer operations, which consist of Clifford gates and preparation and measurement in the standard basis, is efficiently simulable on a classical computer [86, 87].

In order to construct a universal quantum computer it is sufficient to include an arbitrary non-Clifford gate to the set of Clifford gates such that the order of the qudits is a prime number. A proof of this result can be found in appendix D of [88], whereby state injection is shown to convert distilled magic states into non-Clifford gates.

We obtain a new family of universal quantum gate sets through the inclusion of any non-Clifford gate to the generating set of the n-qudit Clifford group, where the order of the qudits d is a prime,

$$\{\text{non-Clifford gate}, X, Y, Z, F, G, \mathfrak{F}_{\mathrm{s}}\}. \qquad (103)$$

Some examples of non-Clifford gates include the Toffoli gate and $\pi/8$ gate [51].

An alternative construction of a universal quantum computer is derived from the entangling property of $\mathfrak{F}_{\mathrm{s}}$ on 2-qudits. The work of Bremner et al. [89] and Brylinski and Brylinski [90] demonstrated that the inclusion of any 2-qudit entangling gate to the set of all single qudit gates is universal for quantum computation. Therefore we obtain the following universal gate set by including $\mathfrak{F}_{\mathrm{s}}$ to the set of single qudit gates,

$$\{\text{single qudit gates}, \mathfrak{F}_{\mathrm{s}}\}. \qquad (104)$$

The quantum gate sets (103) and (104) are new to quantum computation, and the string Fourier transform is common to both sets. Combining the universal gate sets that arise from the string Fourier transform with our diagrams for qudits, transformations, and measurements constitutes a new model of universal quantum computation, that we call the PAPPA model.

In other words holographic software has enough structure to efficiently simulate any other model of universal quantum computation by the modified Church-Turing thesis in [5]. It would be interesting to study the intersimulatability between the known universal quantum gate sets and (103) and (104). Furthermore, the universality arising from the SFT implies that the PAPPA model provides a new realization of universal quantum simulation [60, 63].

The pioneering work of Jozsa and Linden on the connections between exponential computational speed-ups and multipartite entanglement [91] provides further motivation to explore the role of the string Fourier transform in quantum algorithms. In the problem of simulating physical phenomena it would be interesting to realize the string Fourier transform with a small-scale quantum simulator to search for the first example of a quantum advantage. See §VII of [59] for some applications of quantum simulation.

VI. CONCLUSION

We relate holographic software to communication. We have given a comprehensive dictionary to translate back and forth between algebraic protocols and diagrammatic software. We found new protocols in this way.

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[1] R. Penrose, Application of Negative Dimension Tensors, pp. 221–244, in "Combinatorial Mathematics and its Applications," Edited by D.J.A. Welsh, Academic Press, 1971.

[2] S. Abramsky and B. Coecke. A categorical semantics of quantum protocols, *Logic in Computer Science, 2004. Proceedings of the 19th Annual IEEE Symposium* IEEE (2004).

[3] B. Coecke and A. Kissinger, Picturing quantum processes: A first course in quantum theory and diagrammatic reasoning, Cambridge University Press, to be published.

[4] A. Kitaev, Fault-tolerant quantum computation by anyons, *Ann. Phys.* **303** (2003), 2–30, `https://arxiv.org/abs/quant-ph/9707021`, doi:10.1016/S0003-4916(02)00018-0.

[5] M. H. Freedman, A. Kitaev, M. J. Larsen, and Z. Wang, Topological quantum computation, *Bulletin of the American Mathematical Society* Volume 40, Number 1, (2002), 31–38, doi:10.1090/S0273-0979-02-00964-3.

[6] M. H. Freedman, A. Kitaev, and Z. Wang, Simulation of topological field theories by quantum computers, *Commun. Math. Phys.*, **227** (2002), 587–603, doi:10.1007/s002200200635.

[7] L. Kauffman and S. Lomonaco Jr., Quantum entanglement and topological entanglement, *New J. Phys.* **4** (2002) 73, doi:10.1088/1367-2630/4/1/373.

[8] L. Kauffman and S. Lomonaco Jr., Braiding operators are universal quantum gates, *New J. Phys.* **6** (2004) 134, doi:10.1088/1367-2630/6/1/134.

[9] R. Ogburn and J. Preskill, Topological quantum computation, *Quantum computing and quantum communications*, **1509** (1999), 341–356, doi:10.1007/3-540-49208-9˙31.

[10] H. Bombin and M. A. Martin-Delgado, Topological computation without braiding, *Phys. Rev. Lett.* **B 98**, (2007) 160502, doi:10.1103/PhysRevLett.98.160502.

[11] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das, Non-Abelian anyons and topological quantum computation, *Rev. Mod. Phys.* **B 80**, (2008) 1083, doi:10.1103/RevModPhys.80.1083.

[12] M. Levin and X.-G. Wen String-net condensation: A physical mechanism for topological phases, *Phys. Rev. B* **71** (2005) 045110. doi:10.1103/PhysRevB.71.045110.

[13] Yu. Manin, Computable and uncomputable (book in Russian). Moscow, Sovetskoye Radio, 1980.

[14] R. Feynman, Simulating physics with computers, *International Journal of Theoretical Physics* **21**, (1982), Issue 6, 467–488, doi:10.1007/BF02650179.

[15] Yu. Manin, Classical computing, quantum computing, and Shor's factoring algorithm, Séminaire Bourbaki, no. 862 (June 1999), *Astérisque*, vol 266, (2000), 375–404, `https://arxiv.org/abs/quant-ph/9903008`.

[16] A. M. Jaffe and Z. Liu, Planar para algebras, reflection positivity, preprint, `http://arxiv.org/abs/1602.02662`.

[17] J-W. Pan, Quantum science satellite, *Chin. J. Space Sci.* **34** (2014), 547–549, doi:10.11728/cjss2014.05.547.

[18] C. Biever, China's quantum space pioneer: We need to explore the unknown, *Nature*, doi:10.1038/nature.2016.19166.

[19] J. Chin, *Wall Street Journal*, China's Latest Leap Forward Isn't Just Great—It's Quantum, August 20, 2016, *Link*.

[20] D. M. Greenberger, M. A. Horne, and A. Zeilinger, Going beyond Bell's theorem, `http://arxiv.org/abs/0712.0921` in *Bell's theorem, quantum theory, and conceptions of the universe*, M. Kafakos, editor, Vol. 37 of "Fundamental Theories of Physics," Springer Verlag, Heidelberg (1989), doi:10.1007/978-94-017-0849-4.

[21] M. Aspelmeyer, T. Jennewein, M. Pfennigbauer, W. Leeb, A. Zeilinger, Long-Distance Quantum Communication with Entangled Photons using Satellites, *IEEE Journal of Selected Topics in Quantum Electronics, special issue on "Quantum Internet Technologies"*, **9**, (2003)

1541–1551, doi:10.1109/JSTQE.2003.820918.

[22] J. Yin, . . . , Y.-A. Chen, C.-Z. Peng, and J.-W. Pan, Quantum teleportation and entanglement distribution over 100-kilometre free-space channels, *Nature* **488**, (2012) 185–188, doi:10.1038/nature11332.

[23] X.-S. Ma, . . . , and A. Zeilinger, Quantum teleportation over 143 kilometres using active feed-forward, *Nature* **489**, (2012) 269–273, doi:10.1038/nature11472.

[24] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wooters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen Channels, *Phys. Rev. Lett.* **70**, (1993) 1895, doi:10.1103/PhysRevLett.70.1895.

[25] G. Vallone, D. Bacco,D. Dequal, S. Gaiarin, V. Luceri, G. Bianco, and P. Villores, Experimental Satellite Quantum Communications, *Phys. Rev. Lett.* **115**, (2015) 040502, http://journals.aps.org/prl/abstract/10.1103/PhysRevLett.115.040502.

[26] Z. Tang, R. Chandrasekara, Y.-C. Tan, C. Cheng, L. Sha, G. C. Hiang, D. K. L. Oi, and A. Ling, Generation and Analysis of Correlated Pairs of Photons aboard a Nanosatellite, *Phys. Rev. Appl.* **5**, (2016) 054022, http://journals.aps.org/prapplied/abstract/10.1103/PhysRevApplied.5.054022.

[27] P. Shor, Algorithms for quantum computation: discrete logarithms and factoring, *Foundations of Computer Science, 1994 Proceedings, 35th Annual IEEE Symposium*, doi:10.1109/SFCS.1994.365700.

[28] V. F. R. Jones, Index for subfactors, *Invent. Math.* **72** (1983), 1–25, http://link.springer.com/article/10.1007%2FBF01389127?LI=true.

[29] V. F. R. Jones, A polynomial invariant for links via von Neumann algebras, *Bull. Amer. Math. Soc.* **12** (1985), 103–111, http://www.ams.org/journals/bull/1985-12-01/S0273-0979-1985-15304-2/S0273-0979-1985-15304-2.pdf.

[30] V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials, *Ann. of Math* **126** (1987), no. 2, 335–388, doi:10.2307/1971403.

[31] E. Witten, Topological quantum field theory, *Comm. Math. Phys.* **117** (1988), no. 3, 353–386. doi:10.1007/BF01223371

[32] M. F. Atiyah, Topological quantum field theories, Publications Mathématiques de l'IHÉS **68** (1988), 175–186, doi:10.1007/BF02698547.

[33] N. Reshetikhin and V. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* **103** (1991), 547–597, doi:10.1007/BF01239527

[34] V. Turaev and O. Ya. Viro, State sum invariants of 3-manifolds and quantum $6j$-symbols, *Topology* **31** (1992), 865–902, doi:10.1016/0040-9383(92)90015-A

[35] D. Deutsch, Quantum computational networks, *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 425, No. 1868 (1989), 73–90, doi:10.1098/rspa.1989.0099.

[36] B. Coecke, Quantum picturalism, *Contemporary Physics*, **51:1** (2010), 59–83, http://www.tandfonline.com/doi/pdf/10.1080/00107510903257624

[37] A. M. Jaffe, Z. Liu, and A. Wozniakowski, Compressed teleportation, https://arxiv.org/abs/1605.00321.

[38] A. Ocneanu, *Quantized groups, string algebras and Galois theory for algebras*, Operator algebras and applications, Vol. 2, London Math. Soc. Lecture Note Ser., vol. 136, Cambridge Univ. Press, Cambridge, 1988, pp. 119–172.

[39] S. Bose, V. Vedral, and P. L. Knight, Multiparticle generalization of entanglement swapping, *Phys. Rev. A* **57**, (1998) 822–829, doi:10.1103/PhysRevA.60.194.

[40] H. J. Kimble, The quantum internet, *Nature* **453** (2008), 1023–1030, doi:10.1038/nature07127.

[41] M. Żukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, "Event-ready-detectors" Bell experiment via entanglement swapping, *Phys. Rev. Lett.* **71**, (1993) 4287–4290, doi:10.1103/PhysRevLett.71.4287.

[42] M. A. Nielsen and I. L. Chuang, Programmable quantum gate arrays, *Phys. Rev. Lett.* **79**, (1997) 321, doi:10.1103/PhysRevLett.79.321.

[43] A. Sørensen and K. Mølmer, Error-free quantum communication through noisy channels, *Phys. Rev. A.* **58**, (1998) 2745, doi:10.1103/PhysRevA.58.2745.

[44] D. Gottesman and I. L. Chuang, Demonstrating the viability of universal quantum computation using teleportation and single-qubit operations, *Nature* **402**, (1999) 390–393, doi:10.1038/46503.

[45] P. van Loock and S. L. Braunstein, Multipartite entanglement for continuous variables: a quantum teleportation network, *Phys. Rev. Lett.* **84**, (2000) 3482, doi:10.1103/PhysRevLett.84.3482.

[46] X. Zhou, D. W. Leung, and I. L. Chuang, Methodology for quantum logic gate construction, *Phys. Rev.* **A 62**, (2000) 052316, doi:10.1103/PhysRevA.62.052316.

[47] J. Eisert, K. Jacobs, P. Papadopoulos, and M. B. Plenio, Optimal local implementation of nonlocal quantum gates, *Phys. Rev.* **A 62**, (2000) 052317, doi:10.1103/PhysRevA.62.052317.

[48] S. F. Huelga, J. A. Vaccaro, A. Chefles, and M. B. Plenio, Quantum remote control: teleportation of unitary operations, *Phys. Rev.* **A 63**, (2001) 042303, doi:10.1103/PhysRevA.63.042303.

[49] B. Reznik, Y. Aharonov, and B. Groisman, Remote operations and interactions for systems of arbitrary-dimensional hilbert space: state-operator approach, *Phys. Rev.* **A 65**, (2002) 032312, doi:10.1103/PhysRevA.65.032312.

[50] N. B. Zhao and A. M. Wang, Hybrid protocol of remote implementations of quantum operations, *Phys. Rev.* **A 76**, (2007) 062317, doi:10.1103/PhysRevA.76.062317.

[51] M. A. Nielsen and I. L. Chuang, Quantum computation and quantum information, Cambridge University Press, 10th Anniversary Edition, 2010.

[52] L. Yu, R. B. Griffiths, and S. M. Cohen, Efficient implementation of bipartite nonlocal unitary gates using prior entanglement and classical communication, *Phys. Rev.* **A 81**, (2010) 062315, doi:10.1103/PhysRevA.81.062315.

[53] S. Luo and A. M. Wang, Remote implementations of partially unkown quantum operations and its entanglement costs, (2013), http://arxiv.org/pdf/1301.5866.pdf.

[54] R. Van Meter, Quantum Networking, John Wiley & Sons, 2014.

[55] S. Hu, W.-X. Cui, D.-Y. Wang, C.-H. Bai, Q. Guo, H.-F. Wang, A.-D. Zhu, and S. Zhang, Teleportation of a Toffoli gate among distant solid-state qubits with quantum dots embedded in optical microcavities, *Nature: Scientific Reports* **5** (2015) 11321, doi:10.1038/srep11321.

[56] A. Hutter and D. Loss, Quantum computing with parafermions, *Phys. Rev.* **B 93**, (2016) 125105 1–7, doi:10.1103/PhysRevB.93.125105.

[57] E. Candès, J. Romberg, and T. Tao, Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, *IEEE Transactions on Information Theory* **52** (2006), 489–509. doi:10.1109/TIT.2005.862083

[58] J. Ignacio and P. Zoller, Goal and opportunities in quantum simulation, *Nature Physics* **8** (2012) 264–266, doi:10.1038/nphys2275.

[59] I. M. Georgescu, S. Ashhab, and F. Nori, Quantum simulation, *Rev. Mod. Phys.* **86** (2014) 153, doi:10.1103/RevModPhys.86.153.

[60] S. Lloyd, Universal quantum simulators, *Science* **273** (1996) 1073–1078, doi:10.1126/science.273.5278.1073.

[61] C. Zalka, Efficient simulation of quantum systems by quantum computers, *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 454, No. 1969 (1998), doi:10.1098/rspa.1998.0162.

[62] D. Abrams and S. Lloyd, Simulation of many-body fermi systems on a universal quantum computer, *Phys. Rev. Lett.* **79** (1997) 2586, doi:10.1103/PhysRevLett.79.2586.

[63] R. Somma, G. Ortiz, J. E. Gubernatis, E. Knill, and R. Laflamme, Simulating physical phenomena by quantum networks, *Phys. Rev.* **A 65** (2002) 042323, doi:10.1103/PhysRevA.65.042323.

[64] I. Buluta and F. Nori, Quantum simulators, *Science* **326** (2009) 108–111, doi:10.1126/science.1177838.

[65] A. Trabesinger, Quantum simulation, *Nature Physics* **8** (2012) 263, doi:10.1038/nphys2258.

[66] I. Bloch, J. Dalibard, and S. Nascimbène, Quantum simulations with ultracold quantum gases, *Nature Physics* **8** (2012) 267–276, doi:10.1038/nphys2259.

[67] R. Blatt and C. F. Roos, Quantum simulations with trapped ions, *Nature Physics* **8** (2012) 277–284, doi:10.1038/nphys2252.

[68] A. Aspuru-Guzik and P. Walther, Photonic quantum simulations, *Nature Physics* **8** (2012) 285–291, doi:10.1038/nphys2253.

[69] A. Houck, H. Türeci, and J. Koch, On-chip quantum simulation with superconducting circuits, *Nature Physics* **8** (2012) 292–299, doi:10.1038/nphys2251.

[70] D. Gottesman, The Heisenberg representation of quantum computers, *Talk at International Conference on Group Theoretic Methods in Physics* (1998), `http://arxiv.org/pdf/quant-ph/9807006v1.pdf`.

[71] G. Vidal, Efficient classical simulation of slightly entangled quantum computations, *Phys. Rev. A* **70**, (2004) 052328, doi:10.1103/PhysRevLett.91.147902.

[72] S. Aaronson and D. Gottesman, Improved simulation of stabilizer circuits, *Phys. Rev. Lett.* **91**, (2003) 147902, doi:10.1103/PhysRevA.70.052328.

[73] A. M. Jaffe and F. L. Pedrocchi, Reflection positivity for parafermions, *Commun. Math. Phys.*, **337** (2015), 455–472, doi:10.1007/s00220-015-2340-x.

[74] A. M. Jaffe and B. Janssens, Characterization of reflection positivity, *Commun. Math. Phys.*, (2016), doi:10.1007/s00220-015-2545-z.

[75] V. F. R. Jones, Baxterization, *Inter. J. Modern Physics A* **6** (1991), no. 12, 2035–2043, doi:10.1142/S0217751X91001027.

[76] R. Baxter, Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain I, II, III, *Ann. Phys.* **76** (1973), 1–24, 25–47, 48–71, doi:10.1016/0003-4916(73)90439-9, doi:10.1016/0003-4916(73)90440-5, doi:10.1016/0003-4916(73)90441-7.

[77] C. N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, *Phys. Rev. Lett.*, **19** (1967), 1312–1315, doi:10.1103/PhysRevLett.19.1312.

[78] V. Fateev and A. B. Zamolodchikov, Self-dual solutions of the star-triangle relations in $\mathbb{Z}_N$-models, *Physics Letters* **92A** (1982), 37–39, doi:10.1016/0375-9601(82)90736-8.

[79] J. Fröhlich, New super-selection sectors ("Soliton-States") in two-dimensional Bose quantum field models, *Commun. Math. Phys.* **47** (1976), 269–310, `http://projecteuclid.org/download/pdf_1/euclid.cmp/1103899761`.

[80] J. Fröhlich, Statistics of fields, the Yang-Baxter equation, and the theory of knots and links, in *Non-Perturbative Quantum Field Theory*, G. 't Hooft et al. (eds.) New York, Plenum Press 1988, doi:10.1007/978-1-4613-0729-7˙4.

[81] J. Schliemann, D. Loss, A. H. MacDonald, Double-occupancy errors, adiabaticity, and entanglement of spin qubits in quantum dots, *Phys. Rev. B* **63** (2001), 085311, doi:10.1103/PhysRevB.63.085311.

[82] J. Schliemann, J. Ignacio Cirac, M. Kuś, M. Lewenstein, and D. Loss, Quantum correlations in two-fermion systems, *Phys. Rev. A* **64**, (2001) 022303, doi:10.1103/PhysRevA.64.022303.

[83] C. Jiang, Z. Liu and J. Wu, Noncommutative uncertainty principles, *Jour. Funct. Anal.* **270.1** (2016), 264–311, doi:10.1016/j.jfa.2015.08.007.

[84] J. Preskill, Plug-in quantum software, *Nature* **402** (1999) 357–358, doi:10.1038/46434.

[85] J. M. Farinholt, An ideal characterization of the Clifford operators, *Journal of Physics A: Mathematical and Theoretical*, Vol. 47, No. 30, (2014), 305303, doi:10.1088/1751-8113/47/30/305303.

[86] D. Gottesman, Stabilizer codes and quantum error correction, (1997), `https://arxiv.org/abs/quant-ph/9705052`.

[87] D. Gottesman, Theory of fault-tolerant quantum computation, *Phys. Rev. A* **57**, (1998), doi:10.1103/PhysRevA.57.127.

[88] E. Campbell, H. Anwar, and D. Browne, Magic-state distillation in all prime dimensions using quantum reed-muller codes, *Phys. Rev. X* **2**, (2012), 041021, doi:10.1103/PhysRevX.2.041021.

[89] M. Bremner, C. Dawson, J. Dodd, A. Gilchrist, A. Harrow, D. Mortimer, M. Nielsen, and T. Osborne, Practical scheme for quantum computation with any two-qubit entangling gate, *Phys. Rev. Lett.* **89**, (2002), 247902, doi:10.1103/PhysRevLett.89.247902.

[90] J. L. Brylinski and R. Brylinski, Universal quantum gates, in Mathematics of Quantum Computation, G. Chen and R. K. Brylinski, Editors, Chapman & Hall/CRC, Boca Raton, Florida 2002, `http://arxiv.org/pdf/quant-ph/0108062v1.pdf`.

[91] R. Jozsa and N. Linden, On the role of entanglement in quantum computational speed-up, *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, Vol. 459, No. 2036 (2003), doi:10.1098/rspa.2002.1097.