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The Elliptic Genus and Hidden Symmetry*

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Dedicated to the memory of Harry Lehmann

Abstract: We study the elliptic genus (a partition function) in certain interacting, twist quantum field theories. Without twists, these theories have $N = 2$ supersymmetry. The twists provide a regularization, and also partially break the supersymmetry. In spite of the regularization, one can establish a homotopy of the elliptic genus in a coupling parameter. Our construction relies on *a priori* estimates and other methods from constructive quantum field theory; this mathematical underpinning allows us to justify evaluating the elliptic genus at one endpoint of the homotopy. We obtain a version of Witten's proposed formula for the elliptic genus in terms of classical theta functions. As a consequence, the elliptic genus has a hidden $SL(2, \mathbb{Z})$ symmetry characteristic of conformal theory, even though the underlying theory is not conformal.

1. Introduction

We study coupled complex bosonic and fermionic quantum fields on a two-dimensional space-time cylinder $S^1 \times \mathbb{R}$, where S^1 denotes a circle of length ℓ. The equations are determined by a holomorphic polynomial in n variables called the superpotential,

$$V: \mathbb{C}^n \mapsto \mathbb{C}. \tag{1.1}$$

We denote the degree of this polynomial by

$$\tilde{n} = \text{degree}(V), \quad \text{and we assume } \tilde{n} \geq 2. \tag{1.2}$$

The complex scalar fields φ and the Dirac field ψ have n and $2n$ components respectively,

$$\varphi = \{\varphi_i\}, \quad \text{where } 1 \leq i \leq n, \quad \text{and } \psi = \{\psi_{\alpha,i}\}, \quad \text{where } 1 \leq \alpha \leq 2, \quad 1 \leq i \leq n. \tag{1.3}$$

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In the literature one finds these equations called "Wess–Zumino equations" or sometimes "Landau–Ginzburg equations". For cubic V, the equations reduce to the coupling of a non-linear boson field to the Dirac field by a Yukawa interaction. Hence one occasionally also refers to the equations arising from general V as "generalized Yukawa" equations. In [15, 12] we established the existence of solutions to the Wess–Zumino equations for massive fields. Recently we extended these results by proving the existence of solutions for the equations coupling *massless, multicomponent, twist fields*. The word "twist" refers to the fact that the fields are multi-valued; translation about the spatial circle results in each component of the field being multiplied by a phase. This phase is proportional to a real parameter ϕ that we choose in the interval $\phi \in (0, 2\pi]$, and the periodic case (no twist) corresponds to the limiting value $\phi = 0$. The operators in the field theory act on a Fock–Hilbert space $\mathcal{H}$ over the circle, with domains and other properties of the operators depending on ϕ. For details of these definitions and results see [4, 10, 11].

We study a subset of polynomials V with properties detailed in Sect. 1.1. For these examples, the Hamiltonian $H = H(V)$ is self-adjoint, it is bounded from below, and the heat kernel $e^{-\beta H}$ has a trace for all $\beta > 0$. This semigroup commutes with the translation group generated by the momentum operator P. There is also a $U(1)$ group $U(\theta) = e^{i\theta J}$ of "twist" symmetries of H, where the generator $J = J(V)$ depends on V, see Sect. 1.1. Denote the fermion number operator by N^f, and let $\Gamma = (-I)^{N^f}$ denote a $\mathbb{Z}_2$-grading. In our examples, all four operators H, P, J, and Γ are self-adjoint and mutually commute. Hence the operator $A = \Gamma\, e^{-i\theta J - i\sigma P}$ is unitary, and the operator $A\, e^{-\beta H} = \Gamma\, e^{-i\theta J - i\sigma P - \beta H}$ has a trace for all $\beta > 0$.

The elliptic genus is the partition function

$$\mathfrak{Z}^V = \mathrm{Tr}_{\mathcal{H}} \left(\Gamma e^{-i\theta J - i\sigma P - \beta H} \right). \tag{1.4}$$

In a seminal paper [21], Witten suggested that one could calculate the elliptic genus of these examples in closed form. He gave a proposed formula (for $\phi = 0$) based on an argument that $\mathfrak{Z}^{\lambda V}$ should be independent of a parameter λ, and an "evaluation" of $\mathfrak{Z}$ for $V = 0$. Kawai, Yamada, and Yang [17] elaborated on the algebraic aspects Witten's work and made contact with related proposals of Vafa [19]. From a mathematical point of view, these insights are not definitive; the representation (1.4) is ill-defined if both $V = 0$ and $\phi = 0$, as $e^{-\beta H}$ does not have a trace, and the evaluation is only suggestive. Furthermore, establishing the existence and continuity of $\mathfrak{Z}^{\lambda V}$ requires extensive analysis, beyond the scope of earlier work.

We introduce a regularized $\mathfrak{Z}^V$, with two regularizing parameters. The first regularization mollifies the zero-frequency modes, and enters through the non-zero twisting parameter ϕ, as explained in Sect. 1.1. The second regularization mollifies the high-frequency modes. We denote the regularization parameter by Λ, and we discuss it in detail in Sect. 5 when we give an explicit expression for the supercharge as a densely defined sesqui-linear form on the Hilbert space $\mathcal{H}$. The regularized supercharges determine self- adjoint operators. The elliptic genus depends on the parameter ϕ, and has a regular limit as $\phi \to 0$. (In fact, the genus continues holomorphically to all $\phi \in \mathbb{C}$.) The genus does not depend on the high-frequency mollifier Λ.

Our goal in this paper is to find and exploit infra-red and ultra- violet regularizations that yield all the following:

- a self-adjoint Hamiltonian H that is bounded from below, with a trace class heat kernel,
- the two-parameter group of Lie symmetries of H generated by J and P, and

- a sufficient number of invariant supercharges to study and to compute the elliptic genus.

The method that we use in this paper has many advantages. We use twists to provide the infra-red regularization, and a ultra-violet regularization with the property of *slow decrease at infinity* to provide a non-local cutoff in the Hamiltonian. This regularized Hamiltonian has a form that allows us to establish stability and self-adjointness, as well as the existence of a trace for the heat kernel. This trace is uniform in the ultra-violet regularization parameter Λ, but diverges as the twist parameter $\phi \to 0$. This regularization leaves us with half the number of translation- invariant supercharges that one expects in a twist-free theory. These supercharges also commute with J.

On the other hand, more straightforward regularizations cause difficulty in at least one of these areas, either producing a heat kernel with continuous spectrum, destroying the $e^{i\theta J}$ symmetry that one needs to study the elliptic genus, breaking all supersymmetries, making it impractical to establish stability, or producing error terms in the supersymmetry algebra that elude estimation. For example, introducing a bosonic mass, without a corresponding fermionic mass, provides an infra-red regularization compatible with a trace-class heat kernel and with J-symmetry; but all supersymmetries will be broken. We used this method in [9] to study the quantum-mechanics version of the present problem. As a result, the mathematical analysis became quite lengthy – even in the case of a finite number of degrees of freedom. On the other hand, introducing a mass in both the boson and the fermion destroys the J-symmetry of the Hamiltonian, as well as of all supercharges, requiring the analysis of o ther types of error estimates. Furthermore, a sharp upper momentum cutoff in the interaction produces non-localities that defy estimation.

One new ingredient in our program is to generalize the framework of constructive quantum field theory to cover twist fields. We carry this out in more detail in [10]. A second new ingredient involves identifying and studying cancellations that occur in the geometric invariants we study, and we give the details of these cancellations. We begin in Sect. 5 with operator estimates, that justify representations of the invariants by invariants of a sequence of approximating problems. Related estimates show that we can exhibit cancellations in the difference quotients for the approximating problems. In order to estimate these cancellations, we pass from operator estimates to the study of traces in Sect. 6 and Sect. 7.

Twisting *partially* breaks supersymmetry, as explained in detail in [11]. Half the supercharges are translation and twist invariant, while the other half of the supercharges are not. The elliptic genus can be written as a function of the invariant charges. We restrict the $3n$-real twisting angles to lie on a line in $\mathbb{R}^{3n}$, parameterized by one angle ϕ. Doing this yields one invariant supercharge that we denote Q, and which commutes both with translations and the twist group. This supercharge satisfies $Q^2 = H + P$. A second supercharge (one that formally exists for $\phi = 0$) is neither translation nor twist invariant. But it is well-behaved in the sense that we can estimate the error terms in the supersymmetry algebra, and we use one of these estimates in this paper.[1]

In the end, we obtain the representation of the elliptic genus in terms of theta functions. The partition function then satisfies certain properties under transformations defined by

[1] Other estimates on the error terms in the supersymmetry algebra play a role if one wants to identify the limiting quantum field theory with full supersymmetry in the limit as the twists are removed. The elliptic genus turns out to be the boundary value of an entire function of $\phi \in \mathbb{C}$. In particular, the limit $\phi \to 0$ exists. Since the Hilbert space and operators we study depend on ϕ, we define a limit of field theories as a limit of expectation values. With such a limit, as long as we keep a well-behaved, non-zero potential, we recover a standard quantum field theory as $\phi \to 0$.

the modular group $SL(2,\mathbb{Z})$, acting on the complex space-time coordinate τ defined below. At first this seems surprising, as the theta functions and conformal symmetry are generally associated with zero mass fields or with conformal field theory. For this reason, we describe the $SL(2,\mathbb{Z})$ symmetry as a *hidden* aspect of these Wess–Zumino models.

Our results here build on work of Witten [21] and Connes [1], combining these ideas with results from our theory of twist quantum fields [4,10] and our work in [6]. The elliptic genus is an index invariant, and as explained in Sect. IX of [6], it fits into the general framework of equivariant, non-commutative geometry (entire cyclic cohomology), characterized by the Dirac operator Q on loop space. However, the elliptic genus is only one such invariant, from a whole family of invariants, that result from the JLO-cocycle [13]. Therefore we suggest that it may be possible, within the framework of the Wess–Zumino examples that we study here, to find closed form expressions for some other invariants given in [6]. We formulated various representations for such invariants in [7,9], and these might be useful in computation.

We prove here the representation for the elliptic genus $\mathfrak{Z}^V$. Our proof relies on a series of *a priori* estimates and other methods from constructive quantum field theory. In particular, we study $\mathfrak{Z}^{\lambda V}$, where λ denotes a real parameter, and establish differentiability of $\mathfrak{Z}^{\lambda V}$ in λ for $\lambda > 0$, and eventually that $\mathfrak{Z}^{\lambda V}$ is a constant function of λ. Another key estimate is to show that $\mathfrak{Z}^{\lambda V}$ does not jump at $\lambda = 0$. In fact, $\mathfrak{Z}^{\lambda V}$ is *a priori* Hölder continuous at $\lambda = 0$. We obtain any positive Hölder exponent $\alpha < 2/(\tilde{n}-1)$, namely there is a constant $M = M(\alpha, V, \Lambda)$ such that

$$\left| \mathfrak{Z}^{\lambda V} - \mathfrak{Z}^{0} \right| \leq M\,\lambda^{\alpha}, \tag{1.5}$$

for $\lambda \in [0,1]$. For potentials of large degree this exponent is small, but strictly positive. These two results combine with the vanishing of the derivative, to show that $\mathfrak{Z}^{\lambda V}$ is actually a constant function of $\lambda \in [0,1]$. We then compute $\mathfrak{Z}^V$ by evaluating $\mathfrak{Z}^0$.

1.1. Assumptions. Let us give more details. The real-time bosonic field $\varphi_{\rm RT} = \{\varphi_{{\rm RT},i}\}$ has n components designated $\varphi_{{\rm RT},i}$, where $1 \leq i \leq n$. The corresponding real-time fermionic fields $\psi_{\rm RT} = \{\psi_{{\rm RT},\alpha,i}\}$ has $2n$ components labeled by α, i with i as before and $1 \leq \alpha \leq 2$. All these fields are complex, and so given $3n$ twist constants $\Omega = \{\Omega_i^b, \Omega_{\alpha,i}^f\}$, there is a one-parameter group $U(\theta)$ such that

$$U(\theta)\varphi_{{\rm RT},i}U(\theta)^* = e^{i\Omega_i^b\theta}\varphi_{{\rm RT},i}, \quad \text{and} \quad U(\theta)\psi_{{\rm RT},\alpha,i}U(\theta)^* = e^{\Omega_{\alpha,i}^b\theta}\psi_{{\rm RT},\alpha,i}. \tag{1.6}$$

Also, the momentum operator implements spatial translations,

$$e^{i\sigma P}\varphi_{{\rm RT},i}(x,t)e^{-i\sigma P} = \varphi_{{\rm RT},i}(x-\sigma,t),$$

and

$$e^{i\sigma P}\psi_{{\rm RT},\alpha,i}(x,t)e^{-i\sigma P} = \psi_{{\rm RT},\alpha,i}(x-\sigma,t). \tag{1.7}$$

These properties uniquely determine each generator J and P, up to an additive constant; we choose these constants in the *normalization condition* NC below.

A twist field has the additional property that these two groups are related. Translation around the circle results in multiplying each component of the field by a phase. Thus there are $3n$-independent twisting angles $\chi = \{\chi_i^b, \chi_{\alpha,i}^f\}$ such that

$$\varphi_{{\rm RT},i}(x+\ell,t) = e^{\chi_i^b}\,\varphi_{{\rm RT},i}(x,t), \text{ and } \psi_{{\rm RT},\alpha,i}(x+\ell,t) = e^{\chi_{\alpha,i}^b}\,\psi_{{\rm RT},i}(x,t). \tag{1.8}$$

Our superpotential V is a holomorphic polynomial from $\mathbb{C}^n$ to $\mathbb{C}$, and it determines the coupling of φ_{RT} with ψ_{RT}. Let V_i denote the directional derivative of V, namely $V_i(z) = \partial V(z)/\partial z_i$. We study a holomorphic polynomial superpotential V with two other basic properties: the potential is *quasi-homogeneous* (QH) and the potential satisfies certain *elliptic bounds* (EL). Furthermore, we assume that the twist constants and twisting angles satisfy certain *twist relations* (TR). Finally we assume certain *normalization conditions* (NC). We now briefly summarize these four hypotheses:

QH (Quasi-homogenity) The superpotential function $V: \mathbb{C}^n \mapsto \mathbb{C}$ is a holomorphic, quasi-homogeneous polynomial of degree $\tilde{n}$ at least two. This means that there are n constants Ω_i called *quasi-homogeneous weights*, such that $0 < \Omega_i \leq \frac{1}{2}$ and

$$V(z) = \sum_{i=1}^{n} \Omega_i \, z_i \, \frac{\partial V(z)}{\partial z_i}. \tag{1.9}$$

EL (Elliptic Property) Given $0 < \epsilon$, there exists $M < \infty$ such that the function V satisfies

$$|\partial^\alpha V| \leq \epsilon \, |\partial V|^2 + M, \qquad \text{and} \qquad |z|^2 + |V| \leq M\left(|\partial V|^2 + 1\right). \tag{1.10}$$

Here $\partial^\alpha V$ denotes any multi-derivative of V, while $|z|$ denotes the magnitude of z, and $|\partial V|^2 = \sum_{i=1}^{n} |\partial V/\partial z_i|^2$ is the squared magnitude of the gradient of V.

TR (Twist Relations) Define the $3n$ twist constants Ω in J as functions of the n quasi-homogeneous weights Ω_i ,

$$\Omega_i^b = \Omega_i, \quad \Omega_{1,i}^f = \Omega_i, \qquad \text{and } \Omega_{2,i}^f = 1 - \Omega_i. \tag{1.11}$$

Choose the $3n$ twisting angles χ to be proportional to the twist constants Ω, namely

$$\chi_i^b = \Omega_i \phi, \quad \chi_{1,i}^f = \Omega_i \phi, \qquad \text{and } \chi_{2,i}^f = (1 - \Omega_i)\,\phi, \tag{1.12}$$

where ϕ is a single twisting parameter that we take to lie in the interval $(0, \pi]$.

NC (Normalization Conditions) Choose the additive constants in the generators J and P so the Fock ground state Ω_{vac} is an eigenvector with the following eigenvalues[2]:

$$P\Omega_{\mathrm{vac}} = 0, \qquad \text{and } J\Omega_{\mathrm{vac}} = -\frac{1}{2}\hat{c}\,\Omega_{\mathrm{vac}},$$

where

$$\hat{c} = \sum_{i=1}^{n}\left(\Omega_{2,i}^f - \Omega_{1,i}^f\right) = \sum_{i=1}^{n}(1 - 2\Omega_i). \tag{1.13}$$

This ensures that J and $-J$ have the same spectrum.

In [10] we establish

Proposition 1.1. *Assume that V is a holomorphic polynomial satisfying EL of Sect. 1.1.*

(i) *There exists a self-adjoint quantum field twist Hamiltonian $H(V)$ that is the norm-resolvent limit of a sequence of approximating Hamiltonians $H_\Lambda(V)$ defined in Sect. 4.*

[2] The constant $\hat{c}$ recurs in these problems and is called the *central charge*. In fact $\hat{c}$ characterizes the weight of the elliptic genus as a modular function, as pointed out in [21].

(ii) *The self-adjoint semi-group $e^{-\beta H(V)}$ is trace class for $\beta > 0$.*
(iii) *Suppose in addition that V is quasi-homogeneous, and that the twist constants Ω and the twisting angles χ satisfy TR. Then the Hamiltonians H_Λ and H both commute with the two-parameter unitary group $e^{i\theta J + i\sigma P}$ of space translations and twists, and they also commute with $\Gamma = (-I)^{N^f}$.*

We introduce some further notation. With $\Im(\tau)$ the imaginary part of τ, let $\mathbb{H} = \{\tau : 0 < \Im(\tau)\}$ designate the upper half plane. We use the parameter $\sigma \in \mathbb{R}$, and the strictly positive parameters β, θ, and ϕ. We take

$$\tau = \frac{\sigma + i\beta}{\ell} \in \mathbb{H}. \tag{1.14}$$

In terms of these parameters, define the variables

$$q = e^{2\pi i\tau}, \text{ so } |q| < 1, \quad y = e^{i\theta}, \text{ so } |y| = 1, \quad \text{and } z = e^{i\phi\tau}, \text{ so } |z| < 1. \tag{1.15}$$

Consider partition functions as functions of τ, θ, and ϕ, related to q, y, and z as above. The Jacobi theta function of the first kind $\vartheta_1(\tau, \theta)$, defined for $\tau \in \mathbb{H}$, for $\theta \in \mathbb{C}$, with period 8 in τ, and with period 4π in θ, is given by

$$\vartheta_1(\tau, \theta) = i q^{\frac{1}{8}} \left(y^{-\frac{1}{2}} - y^{\frac{1}{2}} \right) \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n y)(1 - q^n y^{-1}). \tag{1.16}$$

This function is odd in the second variable, namely $\vartheta_1(\tau, \theta) = -\vartheta_1(\tau, -\theta)$. We follow the standard notation in Sect. 21.3 of Whittaker and Watson [20], with the exceptions noted above.

2. Main Results

We study the partition function

$$\mathfrak{Z}^{\lambda V} = \mathrm{Tr}_{\mathcal{H}} \left(\Gamma e^{-i\theta J - i\sigma P - \beta H(\lambda V)} \right). \tag{2.1}$$

For $V = 0$, the heat kernel $e^{-\beta H_0}$ is also trace class, on account of the non-zero twisting parameter ϕ. Given a non-zero potential V satisfying QH and EL, we associate a family of potentials λV, where $\lambda \in [0, 1]$, and also a generator J of symmetry with parameters Ω specified by TW and normalization given by NC. The partition function $\mathfrak{Z}^0$ defined by $\lambda = 0$ has an implicit dependence on V, brought about through the choice of J. We devote the remainder of this paper to establishing the following theorem and its corollary.

Theorem 2.1. *Assume the polynomial potential V of degree $\tilde{n} \geq 2$ satisfies QH and EL of Sect. 1.1. Consider the self-adjoint Hamiltonian $H = H(\lambda V)$, as defined in Proposition 1.1 for $\lambda \geq 0$. Assume that the twist fields satisfy assumptions TR, and that P and J satisfy NC.*

(i) *The map*

$$\lambda \mapsto \mathfrak{Z}^{\lambda V}(\tau, \theta, \phi) \tag{2.2}$$

is differentiable in λ for $\lambda > 0$.

(ii) *Choose* α *so that* $0 \le \alpha < 2/(\tilde{n}-1)$. *There exists a constant* $M = M(\alpha, V)$ *such that for* $\lambda \in [0,1]$,

$$\left| \mathfrak{Z}^0 - \mathfrak{Z}^{\lambda V} \right| \le M \lambda^{\alpha}. \tag{2.3}$$

Corollary 2.2. *The map (2.2) is constant for* $0 \le \lambda \le 1$. *The partition function* $\mathfrak{Z}^V$ *depends on* V *only through its weights* Ω, *and it equals*

$$\mathfrak{Z}^V(\tau,\theta,\phi) = z^{\hat{c}/2} \prod_{i=i}^{n} \frac{\vartheta_1(\tau, (1-\Omega_i)(\theta - \phi\tau))}{\vartheta_1(\tau, \Omega_i(\theta-\phi\tau))}. \tag{2.4}$$

Remark. Corollary 2.2 shows that $\mathfrak{Z}^V(\tau,\theta,\phi)$ extends to a holomorphic function for $\tau \in \mathbb{H}$, $\theta \in \mathbb{C}$, and $\phi \in \mathbb{C}$. If $a,b,c,d \in \mathbb{Z}$, and $ad - bc = 1$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. Let

$$\tau' = \frac{a\tau+b}{c\tau+d}, \qquad \theta' = \frac{\theta}{c\tau+d}, \qquad \text{and} \qquad \phi' = \frac{\phi\tau}{a\tau+b}. \tag{2.5}$$

The analytic continuation of the partition function $\mathfrak{Z}^V(\tau,\theta,\phi)$ obeys the transformation law

$$\mathfrak{Z}^V(\tau',\theta',\phi') = e^{2\pi i \left(\frac{\hat{c}}{8}\right)\left(\frac{c(\theta-\phi\tau)^2}{c\tau+d}\right)} \mathfrak{Z}^V(\tau,\theta,\phi). \tag{2.6}$$

One obtains limiting values from the representation (2.4) as the parameters ϕ, θ, or q vanish; these limits are not uniform and do not commute. Define the integer-valued index of the self-adjoint operator Q with respect to the grading Γ as the difference in the dimension of the kernel and the dimension of the cokernel of Q as a map from the $+1$ eigenspace of Γ to the -1 eigenspace of Γ. Denote this integer by $\mathrm{Index}_{\Gamma}(Q)$.

Corollary 2.3. *We have the following limits.*

(i) *As* ϕ *tends to zero, the partition function converges to*[3]

$$\lim_{\phi\to 0} \mathfrak{Z}^V = \prod_{i=i}^{n} \frac{\vartheta_1(\tau, (1-\Omega_i)\theta)}{\vartheta_1(\tau, \Omega_i \theta)}. \tag{2.7}$$

As $\theta \to 0$, *the partition function converges to*

$$\lim_{\theta\to 0} \mathfrak{Z}^V = z^{\hat{c}/2} \prod_{i=1}^{n} \frac{\vartheta_1(\tau, (1-\Omega_i)\phi\tau)}{\vartheta_1(\tau, \Omega_i \phi\tau)}. \tag{2.8}$$

[3] The existence of a field theory for $\phi = 0$ requires special analysis. For $\lambda \ne 0$, this can be established as a consequence of the assumption EL for V. The field theory is the $\phi \to 0$ limit of the twist field theory, and the elliptic genus of the limiting theory is the limit (2.7). It agrees with the formula proposed in [17]. In the case $\lambda = 0$, the elliptic genus also has a $\phi \to 0$ limit as long as $0 < |\theta| < 2\pi$, but this limit is not the genus of a limiting theory.

(ii) *For $\theta \in (0, \pi)$, we may take the iterated limit as $\phi \to 0$ and then $q \to 0$ to obtain the equivariant, quantum- mechanical index studied in [9],*

$$\lim_{q\to 0}\left(\lim_{\phi\to 0} \mathfrak{Z}^V\right) = \prod_{i=1}^{n} \frac{\sin\left((1-\Omega_i)\theta/2\right)}{\sin\left(\Omega_i\theta/2\right)}. \tag{2.9}$$

(iii) *The integer-valued index* $\mathrm{Index}_\Gamma(Q)$ *can be obtained as*

$$\begin{aligned}\mathrm{Index}_\Gamma(Q) &= \lim_{\theta\to 0}\left(\lim_{\phi\to 0} \mathfrak{Z}^V\right) = \lim_{\phi\to 0}\left(\lim_{\theta\to 0} \mathfrak{Z}^V\right)\\ &= \lim_{\theta\to 0}\left(\lim_{q\to 0}\left(\lim_{\phi\to 0} \mathfrak{Z}^V\right)\right) = \prod_{i=1}^{n}\left(\frac{1}{\Omega_i} - 1\right).\end{aligned} \tag{2.10}$$

(iv) *On the other hand,*

$$\lim_{\theta\to 0}\left(\lim_{q\to 0} \mathfrak{Z}^V\right) = \lim_{q\to 0}\left(\lim_{\theta\to 0} \mathfrak{Z}^V\right) = 1. \tag{2.11}$$

Examples. For any n, if $V(z) = \sum_{i=1}^{n} z_i^{k_i}$, with $2 \le k_i \in \mathbb{Z}$, then V satisfies QH and EL, and

$$\Omega_i = \frac{1}{k_i}, \quad \hat{c} = \sum_{i=1}^{n} \frac{k_i - 2}{k_i}, \text{ and } \quad \mathrm{Index}_\Gamma(Q) = \prod_{i=1}^{n}(k_i - 1). \tag{2.12}$$

For $n = 2$, with $V(z) = z_1^{k_1} + z_1 z_2^{k_2}$, the potential also satisfies QH and EL. In this case,

$$\Omega_1 = \frac{1}{k_1}, \quad \Omega_2 = \frac{k_1 - 1}{k_1 k_2}, \quad \hat{c} = 2\frac{(k_1-1)(k_2-1)}{k_1 k_2},$$

and

$$\mathrm{Index}_\Gamma(Q) = k_1(k_2 - 1) + 1. \tag{2.13}$$

Remark. The integer-valued index (2.10) is stable under a class of perturbations of V that are not necessarily quasi-homogeneous. Briefly, we require that $V = V_1 + V_2$, where V_1 satisfies the hypotheses QH and EL above. While V_2 is a holomorphic polynomial, it is not necessarily quasi-homogeneous. In place of this, we assume that the perturbation V_2 is small with respect to V_1 in the following sense: given $0 < \epsilon$, there exists a constant $M_1 < \infty$ such that for any multi-derivative ∂^α of total degree $|\alpha| \ge 1$,

$$|\partial^\alpha V_2| \le \epsilon |\partial V_1| + M_2. \tag{2.14}$$

3. Supercharge Forms

In this section, we define the supercharge Q as a densely-defined, symmetric, sesquilinear form. In later sections, we consider a family of self-adjoint operators Q_Λ that are mollifications of Q. The operators Q_Λ have a norm resolvent limit, showing that the sesquilinear form Q actually defines an unbounded operator. The definition of Q does not require renormalization.

The Hilbert space of our example is a Fock space $\mathcal{H} = \mathcal{H}^b \otimes \mathcal{H}^f$. The bosonic Hilbert space $\mathcal{H}^b$ and the fermionic Hilbert space $\mathcal{H}^f$ are the symmetric and respectively the skew-symmetric tensor algebras over the one particle space $\mathcal{K}$. Here $\mathcal{K}$ is the direct sum of $2n$- copies of $L^2(S^1)$. The free Hamiltonian H_0, the momentum operator P, the total number operator $N = N^b + N^f$, and twist generator $J = J(\Omega)$ are self-adjoint operators on $\mathcal{H}$. Here N^b is the total bosonic number operator, and it acts on $\mathcal{H} = \mathcal{H}^b \otimes \mathcal{H}^f$ as $N^b \otimes I$, etc. The bosonic time-zero field $\varphi(x)$, its conjugate field $\pi(x)$ and fermion time-zero fields $\psi(x)$ are operator valued distributions on $\mathcal{H}$.

There is a dense linear subset $\mathcal{D} \subset \mathcal{H}$, obtained by replacing $L^2(S^1)$ by $C_0^\infty(S^1)$, and by taking vectors with a finite number of particles. The domain $\mathcal{D}$ provides a natural domain on which to define operators, and then to extend them by closure. Furthermore the operators N, Γ, H_0, P, and $e^{i\theta J}$ all map $\mathcal{D}$ into $\mathcal{D}$.

In addition to defining operators with the domain $\mathcal{D}$, we also define sesqui-linear forms with domain $\mathcal{D} \times \mathcal{D}$. These are maps from pairs of vectors in $\mathcal{D}$ to $\mathbb{C}$, that are anti-linear in the first vector and linear in the second vector. By polarization, each such form can be expressed as a sum of four diagonal elements, namely as a sum of four expectations in vectors in $\mathcal{D}$. On the domain $\mathcal{D} \times \mathcal{D}$, the components of the time-zero fields $\varphi_i(x)$, $\pi_i(x)$ and $\psi_{\alpha,i}(x)$, as well as normal-ordered polynomials in these components, are sesqui-linear forms; see for example [2]. The values of these forms defined in this way are C^∞ functions of x. We call them C^∞-sesqui-linear forms with the domain $\mathcal{D} \times \mathcal{D}$. Unless we specify otherwise, we use these domains and then extend the resulting operators or forms by closure. Ultimately our goal is to redefine operators and forms with domains determined by the range of a heat kernel of the Hamiltonian.

Choose a potential function V satisfying QH and EL. This potential as a function of the scalar complex, boson field $\varphi(x)$ determines the energy density of our system as follows. Let $\psi(x)$ denote our Dirac field. Monomials in the components of the scalar field $\varphi_i(x)$ (or in the components of the adjoint field, but not simultaneously in the components of the field and of its adjoint) are normal ordered. Since the boson fields and the Dirac fields act on different factors in the tensor product, the product of a normal ordered boson field and a Dirac field is also normal ordered. Let λ denote a real parameter lying in the interval [0, 1]. Define the normal ordered density $D(\lambda; x)$ as the C^∞ sesqui-linear form

$$D(\lambda; x) = \sum_{j=1}^{n} \left\{ i\psi_{1,j}(x) \left(\pi_j(x) - \partial_x \varphi_j(x)^* \right) + \lambda \psi_{2,j}(x) V_j(\varphi(x))^* \right\}, \tag{3.1}$$

with domain $\mathcal{D} \times \mathcal{D}$. The adjoint of a C^∞ sesqui-linear form is also a C^∞ sesqui-linear form. Define the sesqui-linear form $D(\lambda; x)^*$ by polarization of the expectations $\langle f, D(\lambda; x)^* f \rangle = \langle f, D(\lambda; x) f \rangle^*$, for $f \in \mathcal{D}$.

Define the supercharge density $Q(\lambda; x)$ as the sesqui-linear form

$$Q(\lambda; x) = D(\lambda; x) + D(\lambda; x)^*. \tag{3.2}$$

The integral of these densities over S^1 yield supercharges that are densely-defined, sesqui-linear forms with the domain $\mathcal{D} \times \mathcal{D}$, namely

$$D(\lambda) = \int_0^\ell D(\lambda; x)\, dx, \quad \text{and} \quad Q(\lambda) = \int_0^\ell Q(\lambda; x)\, dx = D(\lambda) + D(\lambda)^*, \tag{3.3}$$

where $D(\lambda)^* = \int_0^\ell D(\lambda; x)^*\, dx$. If we also assume the twist assumption TW, then these forms have the properties for all $\lambda \in [0, 1]$, all σ, and all θ,

$$\Gamma\, Q(\lambda) = -Q(\lambda)\,\Gamma, \quad e^{i\sigma P}\, Q(\lambda) = -Q(\lambda)\, e^{i\sigma P},$$

and

$$e^{i\theta J}\, Q(\lambda) = -Q(\lambda)\, e^{i\theta J}. \tag{3.4}$$

The supercharge that we denote $Q(\lambda)$, or sometimes $Q(\lambda V)$, is the one that we study most in this paper. Define $D_0 = D(0)$ and $Q_0 = Q(0)$, and define D_I and Q_I so that

$$D(\lambda) = D_0 + \lambda D_I, \text{ and } Q(\lambda) = Q_0 + \lambda\, Q_I. \tag{3.5}$$

The supercharge Q_0 extends by closure to a self-adjoint operator Q_0. (This means that the form obtained by closing Q_0 with the domain $\mathcal{D} \times \mathcal{D}$ uniquely determines a self-adjoint operator that we also name Q_0.) This operator is essentially self adjoint on the domain $\mathcal{D}$, and also maps this domain into itself. Furthermore, Q_0 commutes with the operator P and with the operator J defined for any Ω. The operator Q_0 anticommutes with $\Gamma = (-I)^{N^f}$. The square of the supercharge operator Q_0 has the property

$$Q_0^2 = H_0 + P. \tag{3.6}$$

As Q_0 commutes with P, it follows from (3.6) that Q_0 commutes with H_0. Furthermore, as $P \le H_0$, we have the elementary inequality of forms,

$$\pm Q_0 \le |Q_0| \le \sqrt{2} H_0^{1/2}. \tag{3.7}$$

We also require the second component of the supercharge. This is a sesqui-linear form $Q_2(\lambda)$, defined as the integral $\int_0^\ell Q_2(\lambda; x)\, dx$ of the density $Q_2(\lambda; x) = D_2(\lambda; x) + D_2(\lambda; x)^*$. Here $D_2(\lambda; x)$ is the C^∞-sesqui-linear form

$$D_2(x) = \sum_{j=1}^{n} \left\{ i\psi_{2,j}(x) \left(\pi_j(x)^* + \partial_x \varphi_j(x)\right) + \lambda \psi_{1,j}(x) V_j(\varphi(x)) \right\} e^{-i\phi x/\ell}. \tag{3.8}$$

As with the first component of the supercharge,

$$\Gamma\, Q_2(\lambda) = -Q_2(\lambda)\, \Gamma, \tag{3.9}$$

and we have the decomposition

$$Q_2(\lambda) = Q_{2,0} + \lambda Q_{2,I}, \tag{3.10}$$

where $Q_{2,0}$ and $Q_{2,I}$ are independent of λ. The form $Q_{2,0}$ uniquely determines a self-adjoint operator that we also denote as $Q_{2,0}$.

However, unlike the case of the operator Q_0, the operator $Q_{2,0}$ is neither translation invariant nor twist invariant. Nevertheless, the square of $Q_{2,0}$ is invariant under both these groups. This square equals

$$Q_{2,0}^2 = H_0 - P + \phi \mathcal{R}, \tag{3.11}$$

where

$$\mathcal{R} = -\frac{2}{\ell} \sum_{i=1}^{n} \int_0^\ell :\psi_{2,i}(x)\psi_{2,i}(x)^*: dx. \tag{3.12}$$

Here $:\cdot:$ denotes normal ordering. An explicit representation of $\frac{\ell}{2}\mathcal{R}$ can be given as a difference of two terms, each term being a sum of number operators for a subset of the fermionic modes, see [4]. This ensures, in particular, that

$$\pm\mathcal{R} \le \frac{2}{\ell} N, \tag{3.13}$$

where N denotes the total number operator.

4. Approximating Supercharge Operators

In order to study the properties of the Hamiltonian, we introduce approximating families of supercharge forms $Q_\Lambda(\lambda)$ indexed by a parameter $\Lambda \in [0, \infty]$, and with the property $Q_0(\lambda) = Q_0$, and $Q_\infty(\lambda) = Q(\lambda)$. Let

$$Q_\Lambda(\lambda) = Q_0 + \lambda Q_{I,\Lambda}, \qquad \text{and} \qquad Q_{2,\Lambda}(\lambda) = Q_{2,0} + \lambda Q_{2,I,\Lambda}. \tag{4.1}$$

In [4] we introduce a family of mollifier functions $\kappa^b_{i,\Lambda}$ and $\kappa^f_{\alpha,i,\Lambda}$ for the scalar and Dirac fields respectively. These mollifiers act by convolution, with a particular mollifier for each field component. The mollifiers have an index Λ that specifies a momentum scale for the mollifier, and each mollifier converges to the Dirac measure δ as $\Lambda \to \infty$. We define mollified time-zero fields $\varphi_\Lambda(x)$ and $\psi_\Lambda(x)$ as sesqui-linear forms with components

$$\varphi_{i,\Lambda}(x) = \int_0^\ell \kappa^b_{i,\Lambda}(x-y)\,\varphi_i(y)\,dy, \quad \text{and} \quad \psi_{\alpha,i,\Lambda}(x) = \int_0^\ell \kappa^f_{\alpha,i,\Lambda}(x-y)\,\psi_{\alpha,i}(y)\,dy. \tag{4.2}$$

We apply the mollifiers only to the fields that occur in the terms Q_I and $Q_{2,I}$. These terms are the interaction terms and are proportional to λ; in this way we mollify the boson and the fermion fields symmetrically.

We construct the mollifiers from a single smooth, positive function $\tilde{\kappa}$ as follows. Let

$$\tilde{\kappa}_{\mathrm{sdi}}(k) = \frac{1}{\left(1+k^2\right)^\epsilon}, \tag{4.3}$$

where $0 < \epsilon \le \epsilon(V)$, and where we choose $\epsilon(V)$ sufficiently small. We choose for $\tilde{\kappa}(k)$ any smooth function such that

$$\tilde{\kappa}_{\mathrm{sdi}}(k) \le \tilde{\kappa}(k) \le \tilde{\kappa}(0) = 1. \tag{4.4}$$

The lower bound on $\tilde{\kappa}(k)$ by the strictly positive function $\tilde{\kappa}_{\mathrm{sdi}}(k)$ is the property that we call *slow decrease at infinity* or sdi, and it ensures that $\tilde{\kappa}(k)$ is sufficiently close to being local, i.e. $\tilde{\kappa}(k) = 1$. We introduced this sdi property in [14] in order to establish stability for a purely-bosonic, bi-local interaction. In the supersymmetric case, the mollified Hamiltonian is bilocal and it is therefore natural to use an sdi mollifier. In [10] we establish stability based on these ideas. We represent the trace of the heat kernel of an approximate Hamiltonian as a functional integral. The sdi property allows us to study a partition of unity of function space, and to show on each patch that the bi-local bosonic self-interaction can be bounded by a similar local self-interaction (with a coefficient that depends on the patch). The method is sufficiently robust that we can also estimate the non-local contributions from the ferm ionic determinant. We describe this phenomenon in more detail in Sect. 5.

We define the family of periodic mollifier functions indexed by Λ by the Fourier series

$$\kappa_\Lambda(x) = \frac{1}{\ell} \sum_{k \in \frac{2\pi}{\ell}\mathbb{Z}} \tilde{\kappa}(k/\Lambda)\, e^{-ikx}, \tag{4.5}$$

where the series for κ_Λ converge in the sense of distributions. Each kernel $\kappa_\Lambda(x)$ satisfies

$$\kappa_\Lambda(x+\ell) = \kappa_\Lambda(x). \tag{4.6}$$

Denote by $\mathcal{S}$ the space of C^∞, periodic functions on the circle. Let κ_Λ denote the integral operator $\kappa_\Lambda : \mathcal{S} \to \mathcal{S}$ defined by the integral kernel $\kappa_\Lambda(x, y) = \kappa_\Lambda(x-y)$. In other words, κ_Λ is the operator of convolution by $\kappa_\Lambda(x)$ on $\mathcal{S}$. Given the usual topology on these smooth functions, the adjoint κ_Λ^+ of the operator κ_Λ acts on the dual space of distributions on the circle, defined by $\left(\kappa_\Lambda^+\varphi\right)(f) = \varphi(\kappa_\Lambda f)$. This adjoint is an integral operator with the kernel $\left(\kappa_\Lambda^+\right)(x, y) = \kappa_\Lambda(y-x) = \overline{\kappa_\Lambda(x-y)}$.

Consider the space $\mathcal{S}_i^b = e^{-i\Omega_i\phi x/\ell}\mathcal{S}$ of smooth functions on the circle satisfying the twist relation $f(x+\ell) = e^{-i\Omega_i\phi} f(x)$. These are the test functions for the components of the bosonic, time-zero, twist field. Likewise define the spaces $\mathcal{S}_{1,i}^f = \mathcal{S}_i^b$ and $\mathcal{S}_{2,i}^f = e^{-i(1-\Omega_i)\phi x/\ell}\mathcal{S}$. For each Λ, define operators $\kappa_{i,\Lambda}^b$ acting on $\left(\mathcal{S}_i^b\right)'$, operators $\kappa_{1,i,\Lambda}^f$ acting on $\left(\mathcal{S}_{1,i}^f\right)'$, and operators $\kappa_{2,i,\Lambda}^f$ acting on $\left(\mathcal{S}_{1,i}^f\right)'$. To simplify notation we designate the mollifiers acting on the dual space by κ_Λ^b, etc., without the adjoints, defining them as the convolution operators with kernels

$$\kappa_{i,\Lambda}^b(x) = \kappa_{1,i,\Lambda}^f(x) = e^{i\Omega_i\phi x/\ell}\kappa_\Lambda(x), \qquad \text{and} \qquad \kappa_{2,i,\Lambda}^f(x) = e^{i(1-\Omega_i/\ell)\phi x}\kappa_\Lambda(x). \tag{4.7}$$

The kernels satisfy $\kappa_{i,\Lambda}^b(x) = \overline{\kappa_{i,\Lambda}^b(-x)}$, and similarly $\kappa_{\alpha,i,\Lambda}^f(x) = \overline{\kappa_{\alpha,i,\Lambda}^f(-x)}$. They satisfy the twist relations

$$\kappa_{i,\Lambda}^b(x+\ell) = \kappa_{1,i,\Lambda}^f(x+\ell) = e^{i\Omega_i\phi}\kappa_{i,\Lambda}^b(x),$$

and

$$\kappa_{2,i,\Lambda}^f(x+\ell) = e^{i(1-\Omega_i)\phi}\kappa_{2,i,\Lambda}^f(x). \tag{4.8}$$

The operators $\kappa_{i,\Lambda}^b$ converge as $\Lambda \to \infty$ to the identity as operators on $\left(\mathcal{S}_i^b\right)'$, and similarly for $\kappa_{\alpha,i,\Lambda}^f$ on $\left(\mathcal{S}_{\alpha,i}^f\right)'$. Correspondingly, the kernels converge as distributions to a Dirac measure δ,

$$\lim_{\Lambda\to\infty} \kappa_{i,\Lambda}^b(x) = \lim_{\Lambda\to\infty} \kappa_{\alpha,i,\Lambda}^f(x) = \delta(x). \tag{4.9}$$

Also define n families of spatially-dependent kernels $v_{i,\Lambda}(x)$ by the Fourier representations that converge in the sense of distributions,

$$v_{i,\Lambda}(x) = e^{i(1-\Omega_i)x\phi/\ell} \left(\frac{1}{\ell} \sum_{k \in \frac{2\pi}{\ell}\mathbb{Z}} |\hat{\kappa}(k/\Lambda)|^2 e^{-ikx} \right). \tag{4.10}$$

In the sense of distributions,

$$\lim_{\Lambda \to \infty} v_{i,\Lambda}(x) = \delta(x). \tag{4.11}$$

With these definitions, we establish in [10] that the forms Q_Λ and $Q_{2,\Lambda}$ determine self-adjoint operators. The operator Q_Λ have the properties

$$\Gamma\, Q_\Lambda = -Q_\Lambda\, \Gamma, \qquad \text{and} \qquad Q_\Lambda\, e^{i\theta J + i\sigma P} = e^{i\theta J + i\sigma P}\, Q_\Lambda, \tag{4.12}$$

for all real θ, σ. Furthermore the operators $Q_{2,\Lambda}$ satisfy

$$\Gamma\, Q_{2,\Lambda} = -Q_{2,\Lambda}\, \Gamma. \tag{4.13}$$

But these operators $Q_{2,\Lambda}$ do not commute with the group $e^{i\theta J + i\sigma P}$.

The operator Q_Λ satisfies the normal relation of the first component of a supercharge and a Hamiltonian H_Λ,

$$Q_\Lambda^2 = H_\Lambda + P. \tag{4.14}$$

Here the Hamiltonian H_Λ is a perturbation of the free, twist- field Hamiltonian $H_0 = H_0^b + H_0^f$, and has the (non-local) form

$$H_\Lambda = H_\Lambda(\lambda V) = H_0 + \sum_{i=1}^{n} \int_0^\ell dx \int_0^\ell dy \;\; V_i(\varphi_\Lambda(x))^* \, \lambda^2 v_{i,\Lambda}(x-y)\, V_i(\varphi_\Lambda(y)) \\ +\; \lambda \left(Y_\Lambda + {Y_\Lambda}^* \right), \tag{4.15}$$

where the boson-fermion coupling Y_Λ is the generalized Yukawa interaction

$$Y_\Lambda = Y_\Lambda(V) = \sum_{i,i'=1}^{n} \int_0^\ell \psi_{1,i,\Lambda}(x) \psi_{2,i',\Lambda}(x)^* V_{ii'}(\varphi_\Lambda(x))\, dx. \tag{4.16}$$

On account of the positive definite nature of the kernel $\lambda^2 v_{i,\Lambda}(x)$, the bosonic part of H_Λ, namely

$$H_\Lambda^b = H_0^b + \sum_{i=1}^{n} \int_0^\ell dx \int_0^\ell dy \;\; V_i(\varphi_\Lambda(x))^* \, \lambda^2 v_{i,\Lambda}(x-y)\, V_i(\varphi_\Lambda(y)), \tag{4.17}$$

is a sum of positive operators. In fact, the bosonic Hamiltonian can also be written,

$$H^b = H_0^b + \lambda^2\, Q_{I,\Lambda}^2, \tag{4.18}$$

where we note the identity,

$$Q_{I,\Lambda}(V)^2 = \sum_{i=1}^{n} \int_0^\ell dx \int_0^\ell dy \;\; V_i(\varphi_\Lambda(x))^* \, \lambda^2 v_{i,\Lambda}(x-y)\, V_i(\varphi_\Lambda(y)). \qquad (4.19)$$

The bosonic Hamiltonian H_Λ^b is not normal ordered, and unlike $H_\Lambda(\lambda V)$, it has no limit as $\Lambda \to \infty$.

The second family of approximate supercharges $Q_{2,\Lambda}$ are also related to H_Λ. However, their square has an error term in the standard supersymmetry algebra,

$$Q_{2,\Lambda}^2 = H_\Lambda - P + \phi\mathcal{R}, \qquad (4.20)$$

where $\mathcal{R}$ is the same operator that arose when analyzing the square of the free supercharge $Q_{2,0}$. The error term is given in (3.12). We use the following result from [10]:

Proposition 4.1. *Assume the potential V satisfies the assumptions QH, EL, assume the relations TR, and assume the definitions of Q_Λ, $Q_{2,\Lambda}$, H_Λ, and P in Sect. 3 and Sect. 4. Then the forms Q_Λ, $Q_{2,\Lambda}$, H_Λ, and P define self adjoint operators on $\mathcal{H}$. The operators H_Λ are bounded from below. The operators Q_Λ, H_Λ, and P mutually commute, and they also commute with J.*

5. Estimates on Operators

We consider here the basic properties of the Hamiltonian and the supercharges. This leads to consideration of estimates that involve implicit renormalization cancellations. These estimates depend only on the form of the underlying operators H, P, N, etc., and they lead to inequalities of operators or their norms. These estimates do not involve further cancellations of the sort that arise in the proof of estimates on partition functions, that we consider in the following section.

5.1. A Priori Estimates. The results here require certain *a priori* estimates involving the family of Hamiltonians $H_\Lambda = H_\Lambda(\lambda V)$, or the associated family of self-adjoint semigroups $e^{-\beta H_\Lambda(\lambda V)}$ that the $H_\Lambda(\lambda V)$ generate. The proofs of these estimates are lengthy, so we establish them as the central results in the companion paper [10]. These estimates are of utmost importance, so we give an overview by collecting together the necessary statements. Within the context of constructive quantum field theory, the estimates we assume are of a standard nature, though they have not been previously proved in the context of zero-mass (twist) fields that we use here. The operators occurring in this section have been introduced earlier in this paper. For more details about these definitions, see [4]; for analytic details, see [10].

- In case the following inequalities involve β, we take $\beta > 0$. We choose a given, fixed $\phi \in (0, \pi]$, and a given, fixed V satisfying QH and EL of Sect. 1.1, and we define the Hamiltonian with the twist relations TR. The operators in question act on a Fock space $\mathcal{H} = \mathcal{H}(\Omega, \phi)$ depending on the parameters Ω, ϕ. We fix these parameters throughout the approximations in this paper. By convention, we generally do not note the dependence of constants on ϕ, while we generally indicate the dependence on V.

- We require certain estimates that are uniform in Λ, the parameter that designates the high-frequency mollifier. There exist positive, finite constants $M_1 = M_1(V)$, $M_2 = M_2(V)$, and $M = M(\beta, V)$ that are independent of Λ, and of $\lambda \in (0, 1]$, and such that

$$N \le M_1 H_\Lambda(\lambda V) + M_2, \tag{5.1}$$

$$H_0^{1/2} \le M_1\, H_\Lambda(\lambda V) + M_2, \tag{5.2}$$

and

$$\mathrm{Tr}_{\mathcal{H}}\left(e^{-\beta H_\Lambda(\lambda V)}\right) \le M(\beta). \tag{5.3}$$

There exists a self-adjoint $R(\beta) = R(\beta; \lambda)$, that is a semigroup in β and that depends on the parameter λ, and such that

$$\| e^{-\beta H_\Lambda(\lambda V)} - R(\beta) \| \to 0, \quad \text{as} \quad \Lambda \to \infty, \tag{5.4}$$

for each $\lambda \in (0, 1]$.
- We require the following estimate that is *not* uniform in Λ. Given Λ, there exist constants $M_1 = M_1(\Lambda, V)$ and $M_2 = M_2(\Lambda, V)$ such that for all $\lambda \in [0, 1]$,

$$H_0 + \lambda^2\, Q_{I,\Lambda}^2 \le M_1\, H_\Lambda(\lambda V) + M_2. \tag{5.5}$$

Remarks. It is no loss of generality in (5.5) to increase the constants, if necessary, so in addition $1 \le M_1$ and $H_0 + \lambda^2\, Q_{I,\Lambda}^2 + I \le M_1\, H_\Lambda(\lambda V) + M_2$, so $H_0 + \lambda^2\, Q_{I,\Lambda}^2 + I \le M_1\, (H_\Lambda(\lambda V) + M_2)$ as well. We make this assumption.

From the norm convergence of semigroups (5.4), we infer that the limiting semigroup $R(\beta)$ has a self-adjoint generator $H = H(\lambda V)$. This defines the limiting Hamiltonian, and $R(\beta) = e^{-\beta H(\lambda V)}$. The uniform bound on the trace of $e^{-\beta H_\Lambda(\lambda V)}$ ensures that $H(\lambda V)$ is bounded from below,[4] and there exists a constant $M_3 = M_3(\lambda V)$ such that

$$0 \le H(\lambda V) + M_3. \tag{5.6}$$

For this limiting theory, there is a self-adjoint operator $Q = Q(\lambda V)$ that commutes with P and that anticommutes with Γ, for which

$$Q(\lambda V)^2 = H(\lambda V) + P. \tag{5.7}$$

We comment briefly on the mollifier $\tilde{\kappa}(k/\Lambda)$ that we employ, rather than a mollifier, for example, that completely eliminates Fourier modes with large $|k|$. In the latter case, the approximating Hamiltonians have not been proved to be bounded from below. We first studied the special advantages of a mollifer like Λ with *slow decrease at infinity* in [14], where we used this property that expresses "almost-locality", to show that a class of bosonic Hamiltonians are bounded from below. We showed that the normal-ordered, purely-bosonic bilocal Hamiltonian $:H_\Lambda^b:$, with H_Λ^b of the form (4.17), is bounded from below. Specifically, in [14] we treat the case with $n = 1$, with a massive (rather than a massless) unperturbed Hamiltonian $H_{0,m}^b$, and with no twist, $\phi = 0$.

We outline the basic idea of our method in [10] to utilize the slowly decreasing property of the mollifier to prove the estimate (5.3). We begin by representing $\mathrm{Tr}_{\mathcal{H}}\left(e^{-\beta H_\Lambda}\right)$ as

[4] Without good control over convergence, such as the norm-convergence of semigroups that is the case here, a uniform bound like (5.1) or (5.2) on $H_\Lambda(\lambda V)$ is insufficient information to establish a lower bound on $H(\lambda V)$.

a functional integral. This is the functional integral for the normal-ordered purely bosonic actions, multiplied by a regularized Fredholm determinant arising from the expectation in the fermionic modes. We insert an appropriate partition of unity $1 = \sum_{i'=1}^{\infty} \chi_{i'}$ into this integral, thus dividing the integration into a sum of integrals over patches. To obtain an effective bound, we need to replace the non- local bosonic part of the action by a related local term. We do this on each patch, using several things: the positive definite form of the interaction term, the explicit form of the mollifier function $\hat{\kappa}(k/\Lambda)$, in particular its monotonic property and its slowly decreasing character as a function of $|k|$. Using these properties, we bound the bilocal (boson ic) action from below on the patch $\chi_{i'}$. We obtain a lower bound on the bilocal action with the non-local coupling constant $\lambda^2 v_{i,\Lambda}(x-y)$ by a similar local action but with a local, coupling constant of the form $\lambda^2\, \tilde{\kappa}(i'd/\Lambda)^2\, \delta(x-y)$. Here $d+1=\tilde{n}$ denotes the degree of the polynomial V. The coefficient of λ^2 here is $\tilde{\kappa}(i'd/\Lambda)^2$, and this vanishes as $i' \to \infty$ (namely at high momentum). In fact for constant Λ, we have the asymptotics $\lambda^2 \tilde{\kappa}(i'd/\Lambda)^2 \sim \lambda^2 i'^{-2\epsilon}$. We use the local action to estimate further non-local perturbations of lower degree, as well as local perturbations of lower degree, on the patch $\chi_{i'}$. This results in an additive, constant error term $r_{i'}$ that has a magnitude $|r_{i'}| \leq o(1)(\tilde{\kappa}(i'd/\Lambda)^{-2}) \leq o(1)(i')^{2\epsilon}$, which diverges as $i' \to \infty$. The measure $|\chi_{i'}|$ of the set $\chi_{i'}$ satisfies $|\chi_{i'}| \leq e^{-i'^{\epsilon''}}$, where $\epsilon'' = \epsilon''(V) > 0$. This constant is small, and it depends only on the polynomial V. Therefore, fixing V, we can choose $\epsilon(V) \geq \epsilon > 0$ sufficiently small so that the product $e^{|r_{i'}|}|\chi_{i'}|$ is small for large i'. When summed over i' it leads to a finite estimate on the integral. We also use the approximate local bosonic action to estimate the non-local terms arising from the regularized Fredholm determinants. In this fashion we establish the uniform upper bound (5.3) on the trace of the family of approximating heat kernels. The method to establish the remaining bounds is similar.

5.2. Traces. In this section we collect a few general remarks that we use later. The Schatten p-norm of T for operators on $\mathcal{H}$ is defined as

$$\| T \|_p = \left(\mathrm{Tr}_{\mathcal{H}} \left(\left(T^* T \right)^{p/2} \right) \right)^{1/p} .$$

These norms satisfy Hölder's inequalities $\|TS\|_r \leq \|T\|_p \|S\|_q$, where $r = pq/(p+q)$, and $1 \leq r,\ p,\ q$. Furthermore, the trace norm $\|\cdot\|_1$ is also given by $\|T\|_1 = \sup_{\text{unitary } U} |\mathrm{Tr}_{\mathcal{H}}(UT)|$, see Sect. III of [18]. Thus

$$|\mathrm{Tr}_{\mathcal{H}}(T)| \leq \|T\|_1. \tag{5.8}$$

An operator T with $\|T\|_1 < \infty$ is said to be *trace class*, and such operators have a basis-independent trace. A sufficient condition to ensure the cyclicity identity of the trace,

$$\mathrm{Tr}_{\mathcal{H}}(AB) = \mathrm{Tr}_{\mathcal{H}}(BA), \tag{5.9}$$

is that A is trace class and B is bounded.

One says that a self-adjoint semigroup $R(t)$ is Θ-summable if there is a function $M(t) < \infty$ such that $\|R(t)\|_1 < M(t)$ for all $0 < t$. A family of semigroups $R_j(t)$ is *uniformly Θ-summable* if

$$\left\| R_j(t) \right\|_1 \leq M(t), \tag{5.10}$$

for all j.

Proposition 5.1. *Assume that* $\{R_j(t)\}$ *are a family of uniformly* Θ*-summable semigroups on a Hilbert space* $\mathcal{H}$*, and assume that* $\left\|R_j(t) - R(t)\right\| \to 0$ *as* $j \to \infty$*. Then* $R(t)$ *is trace class, and* $R_j(t)$ *converges to* $R(t)$ *in trace norm,*

$$\lim_{j\to\infty} \left\|R_j(t) - R(t)\right\|_1 = 0, \ \textit{and} \ \|R(t)\|_1 \leq M(t), \ \textit{for all} \ 0 < t. \tag{5.11}$$

Furthermore, for any bounded operator A*,*

$$\mathrm{Tr}_{\mathcal{H}}\,(AR(t)) = \lim_{j\to\infty} \mathrm{Tr}_{\mathcal{H}}\left(AR_j(t)\right), \ \textit{for all} \ 0 < t. \tag{5.12}$$

Proof. Write

$$R_j(t) - R_m(t) = R_j(t/2)\left(R_j(t/2) - R_m(t/2)\right) + \left(R_j(t/2) - R_m(t/2)\right) R_m(t/2). \tag{5.13}$$

Thus by Hölder's inequality,

$$\|R_j(t) - R_m(t)\|_1 \leq 2M(t/2)\,\|R_j(t/2) - R_m(t/2)\|. \tag{5.14}$$

Hence $R_j(t)$ is a Cauchy sequence in the Schatten ideal of trace class operators. Thus there exists a trace-class limit $\widetilde{R}(t)$, for which

$$\left\|R_j(t) - \widetilde{R}(t)\right\|_1 \to 0, \ \text{and} \ \left\|\widetilde{R}(t)\right\|_1 \leq M(t). \tag{5.15}$$

Since $\left\|R_j(t) - \widetilde{R}(t)\right\| \leq \left\|R_n(t) - \widetilde{R}(t)\right\|_1$, we infer from (5.15) that $\widetilde{R}(t) = R(t)$.

Since $R(t)$ and $R_j(t)$ are trace class, if A is bounded then $AR(t)$ and $AR_j(t)$ are also trace class. For a trace class operator T, we use (5.8) and Hölder's inequality to obtain

$$\left|\mathrm{Tr}_{\mathcal{H}}\left(AR_j(t) - AR(t)\right)\right| \leq \left\|AR_j(t) - AR(t)\right\|_1 \leq \|A\|\left\|R_j(t) - R(t)\right\|_1, \tag{5.16}$$

from which (5.12) follows. This completes the proof of the lemma. □

Lemma 5.2. *Let* $e^{-\beta H}$ *be a self-adjoint,* Θ*-summable semigroup, and let* A *be a bounded operator on* $\mathcal{H}$*. Then the map*

$$(\sigma, \ \beta) \mapsto \mathrm{Tr}_{\mathcal{H}}\left(A\, e^{i\sigma P - \beta H}\right) \tag{5.17}$$

extends holomorphically in β *to all* $i\beta \in \mathbb{H}$ *(keeping* $\sigma \in \mathbb{R}$ *fixed). Suppose the unitary group* $e^{i\sigma P}$ *is a symmetry of* H*, and there exist constants* $M_1, M_2 < \infty$ *such that*

$$\pm P \leq M_1 H + M_2. \tag{5.18}$$

Then for $i\beta \in \mathbb{H}$*, the map (5.17) extends analytically in* σ *into a strip about the real axis of width proportional to* $\Re(\beta)$*, and otherwise only depending on* M_1 *and* M_2*.*

Proof. Theta summability ensures that H is bounded from below, so it is no loss of generality to add a constant to H so $H \geq I$. With this convention, we can replace (5.18) by the assumption that there exists a constant $M = M(M_1, M_2) < \infty$ such that

$$\pm P \leq M\,H. \tag{5.19}$$

To prove analytic continuation in β, it is sufficient to establish a neighborhood of absolute convergence for the power series in ϵ of

$$\mathrm{Tr}_{\mathcal{H}}\left(e^{-(\beta+\epsilon)H}\right) = \sum_{n=0}^{\infty} (-\epsilon)^n /n!\mathrm{Tr}_{\mathcal{H}}\left(H^n e^{-\beta H}\right),$$

starting initially with real β. Express β in its real and imaginary parts $\beta = \Re(\beta)+i\Im(\beta)$. The operator $e^{e\Im(\beta)}$ is unitary, so for $0 < \Re(\beta)$, the operator $H^n e^{-\beta H/2}$ is bounded in norm by $(n/\Re(\beta))^n$. So using Hölder's inequality and (5.8) $|\mathrm{Tr}_{\mathcal{H}}\left(H^n e^{-\beta H}\right)| \leq (n/\Re(\beta))^n \, \|e^{-\beta H/2}\|_1$. Then the exponential series converges absolutely for $|\epsilon| < \Re(\beta)/e$, yielding

$$\sum_{n=0}^{\infty} \frac{|\epsilon|^n}{n!} |\mathrm{Tr}_{\mathcal{H}}\left(H^n e^{-\beta H}\right)| \leq (1 - |\epsilon| e/\Re(\beta))^{-1} \, \|e^{-\Re(\beta)H/2}\|_1 < \infty, \tag{5.20}$$

as desired. We assume that P and H commute, so we simultaneously diagonalize these operators. We conclude from the spectral representation and (5.19) that $|P|^n \leq M^n H^n$ for non-negative integers n. Proceed as above in the domain $|\epsilon| < \Re(\beta)/Me$, the power series in ϵ for $e^{i(\sigma+\epsilon)P} e^{-\beta H2}$ converges absolutely in operator norm. Using Hölder's inequality and (5.8), it then follows that $\mathrm{Tr}_{\mathcal{H}}\left(e^{i\sigma P-\beta H}\right)$ is real analytic in σ for $i\beta \in \mathbb{H}$, and the proof is complete. □

Proposition 5.3. *Assume quantum twist fields interact, with the nonlinearity determined by a polynomial V as specified above. Assume QH, EL, and TR of Sect. 1.1. Then there exist constants M_1 and M_2, independent of Λ , and such that*

$$\pm P \leq M_1 H_\Lambda + M_2. \tag{5.21}$$

As a consequence, with a new constant M_1,

$$Q_\Lambda^2 \leq M_1 H_\Lambda + M_2. \tag{5.22}$$

Proof. The identity $Q_\Lambda^2 = H_\Lambda + P$ of (4.14) gives an upper bound on $-P$,

$$-P \leq H_\Lambda. \tag{5.23}$$

In order to obtain an upper bound on P, we take into account the details concerning the second component of the supercharge $Q_{2,\Lambda}$. From the relation (4.20) we infer that

$$P \leq H_\Lambda + \phi\mathcal{R}. \tag{5.24}$$

Thus to establish an upper bound on P, it is sufficient to establish an upper bound on $\mathcal{R}$ in terms of H_Λ. We use the explicit form for $\mathcal{R}$ in (3.12), and the following comment; see [4] for details. It therefore follows that $\mathcal{R}$ satisfies the bound

$$\pm\mathcal{R} \leq \frac{2}{\ell} N, \tag{5.25}$$

where N is the total number-of-particles operator. Using (5.1), we infer that $P \leq M_1 H_\Lambda + M_2$, with constants independent of Λ. The bound (5.21) then follows, and from (4.14) we also infer (5.22). □

5.3. Continuity of the Heat Kernel for $\lambda > 0$. We establish Lipshitz continuity, in the trace-norm topology, of the map

$$\lambda \mapsto e^{-\beta H_\Lambda(\lambda V)}, \tag{5.26}$$

from the parameter $\lambda \in (0, 1]$ into the approximating heat kernels. Stated in detail, for each allowed V, each fixed $j < \infty$, and each fixed $\lambda \in (0, 1]$, and for $|\lambda - \lambda'|$ sufficiently small, there exists a constant M such that

$$\left\| e^{-\beta H_\Lambda(\lambda V)} - e^{-\beta H_\Lambda(\lambda' V)} \right\|_1 \leq M\,|\lambda - \lambda'|. \tag{5.27}$$

Unfortunately, the estimates that we have proved for $H_\Lambda(\lambda V)$ are insufficient to show that the map (5.26) is differentiable in λ, and we do not know whether this is true. Also, we do *not* know whether $\mathrm{Tr}_{\mathcal{H}}\left(e^{-H_\Lambda(\lambda V)}\right)$ is differentiable in λ. However, in the next subsection we show that the partition function $\mathrm{Tr}_{\mathcal{H}}\left(\Gamma e^{-H_\Lambda(\lambda V)}\right)$ *is* differentiable in λ.

We study the λ-derivative of the approximating family of heat kernels. For $\lambda, \lambda' \in (0, 1]$, and $\lambda \neq \lambda'$, define the difference quotient of $e^{-\beta H_\Lambda(\lambda V)}$ by

$$\Delta^\beta(\lambda, \lambda') = \frac{e^{-\beta H_\Lambda(\lambda V)} - e^{-\beta H_\Lambda(\lambda' V)}}{\lambda - \lambda'}, \qquad \text{and let} \qquad \lambda_{\min} = \min\{\lambda, \lambda'\}. \tag{5.28}$$

In the following we let $R(\beta)$ denote the self-adjoint, trace- class semigroup generated by $H_\Lambda(\lambda V)$, and let $R'(\beta)$ denote the similar semigroup generated by $H_\Lambda(\lambda' V)$,

$$R(\beta) = e^{-\beta H_\Lambda(\lambda V)}, \qquad \text{and} \qquad R'(\beta) = e^{-\beta H_\Lambda(\lambda' V)}. \tag{5.29}$$

Define the function $F^\beta_\Lambda(\lambda, \lambda', s)$ for $\lambda, \lambda', s \in (0, 1)$ for allowed potentials V by

$$\begin{aligned} &F^\beta_\Lambda(\lambda, \lambda', s)\\ &= -\beta\, e^{-s\beta H_\Lambda(\lambda V)} \left(Q_\Lambda(\lambda V)\, Q_{I,\Lambda}(V) + Q_{I,\Lambda}(V)\, Q_\Lambda(\lambda' V)\right) e^{-(1-s)\beta H_\Lambda(\lambda')}. \end{aligned} \tag{5.30}$$

We also write this as

$$\begin{aligned} &F^\beta_\Lambda(\lambda, \lambda', s)\\ &\quad = -\beta R(s\beta) \left(Q_\Lambda(\lambda V)\, Q_{I,\Lambda}(V) + Q_{I,\Lambda}(V)\, Q_\Lambda(\lambda' V)\right) R'((1-s)\beta). \end{aligned} \tag{5.31}$$

Note that the bound (5.3) ensures that $\Delta^\beta(\lambda, \lambda')$ is trace class. By itself, this does not establish (5.27), as the trace norm may diverge as $\lambda' \to \lambda$. Also the bound (5.3), taken together with the bound (5.5), ensures that $F^\beta_\Lambda(\lambda, \lambda', s)$ is the sum of two trace-class operators. In order to verify that $F^\beta_\Lambda(\lambda, \lambda', s)$ is trace class, write each of the two heat kernels in (5.30) as the square of a heat kernel. The bound (5.3) shows that one of the heat kernel factors by itself is trace class. The second heat kernel multiplies $Q_\Lambda(\lambda V)$, $Q_\Lambda(\lambda' V)$, or $Q_{I,\Lambda}(V)$ (either on the left or on the right); the estimates (5.3) and (5.5) show that each such product is bounded. Since the product of a bounded operator with a trace-class operator is trace class, we infer that $F^\beta_\Lambda(\lambda, \lambda', s)$ is trace class. But we have no control over how the trace-norm diverges (for fixed β) as s approaches an endpoint of the interval. We now address these issues.

Let us denote the degree of the polynomial V by

$$\tilde{n} = \mathrm{degree}(V), \qquad \text{and note} \qquad 2 \leq \tilde{n}, \tag{5.32}$$

in order to satisfy the elliptic growth assumption EL of Sect. 1.1.

Theorem 5.4. *Assume quantum twist fields interact, with the nonlinearity determined by a polynomial* V *as specified above. Assume QH, EL, and TR of Sect. 1.1. Let* $\beta > 0$. *Let* $j \in \mathbb{Z}_+$ *be fixed. Then there exists a constant* $M = M(\beta, \Lambda, V) < \infty$, *such that the difference quotient* $\Delta^\beta(\lambda, \lambda')$ *satisfies the trace-norm bound*

$$\left\| \Delta^\beta(\lambda, \lambda') \right\|_1 \le M\, \lambda_{\min}^{-1+1/(\tilde{n}-1)}, \tag{5.33}$$

for all $\lambda, \lambda' \in (0, 1]$. *Lipshitz continuity (5.27) then follows.*

Theorem 5.4 is contained in Proposition 5.5 and Corollary 5.7 that follow.

Proposition 5.5. *Under the hypotheses of Theorem 5.4, there exists a constant* $M = M(\beta, \Lambda, V) < \infty$ *such that for* $\lambda, \lambda' \in (0, 1]$, *for* $s \in (0, 1)$, *and for* $0 \le \alpha \le 1/(\tilde{n}-1)$, *the following holds:*

(i) *The operator* $F^\beta_\Lambda(\lambda, \lambda', s)$ *defined in (5.30) has a trace norm bounded by*

$$\left\| F^\beta_\Lambda(\lambda, \lambda', s) \right\|_1 \le M\, \lambda_{\min}^{-1+\alpha} \left(s^{-1+\alpha/2} (1-s)^{-1/2} + s^{-1/2} (1-s)^{-1+\alpha/2} \right). \tag{5.34}$$

(ii) *The map* $s \mapsto F^\beta_\Lambda(\lambda, \lambda', s)$ *is continuous in the trace-norm topology.*

Lemma 5.6. *There exists a constant* $M_3 = M_3(j, V)$ *such that the following bounds hold:*

(i) *For any* $\alpha \in [0, 1]$, *the interaction* $Q_{I,\Lambda}(V)$ *satisfies*

$$Q_{I,\Lambda}(V)^{2\alpha} \le M_3^\alpha\, (N + I)^{\alpha(\tilde{n}-1)}$$

and also

$$Q_{I,\Lambda}(V)^{2\alpha} \le M_3^\alpha\, (H_0 + I)^{\alpha(\tilde{n}-1)}\,. \tag{5.35}$$

Here N *is the total number operator and* $\tilde{n}$ *is the degree of* V.

(ii) *The generalized Yukawa interaction* $Y_\Lambda + Y^*_\Lambda = \{Q_0, Q_{I,\Lambda}(V)\}$ *satisfies*

$$\pm\{Q_0, Q_{I,\Lambda}(V)\} \le M_3\, (H_0 + I)^{\tilde{n}-1}\,. \tag{5.36}$$

(iii) *For* $0 \le \alpha \le (\tilde{n}-1)^{-1}$, $0 < \lambda \le 1$, *and* $0 \le \lambda' \le 1$,

$$\left\| (H_\Lambda(\lambda V) + M_2)^{-(1-\alpha)/2}\, Q_{I,\Lambda}(V)\, \left(H_\Lambda(\lambda' V) + M_2 \right)^{-\alpha(\tilde{n}-1)/2} \right\| \le M_3\, \lambda^{-1+\alpha}. \tag{5.37}$$

Proof. The estimates leading to this bound rely on the expansion of the bosonic field into its Fourier representation. The Fourier coefficients of the field are linear in creation and annihilation operators, multiplied by a kernel that is l^2, by virtue of the mollifier $\tilde{\kappa}$, but with an l^2 norm depending on V and also on Λ . These expansions and properties are given in detail in [4]. As a consequence, the operator $Q^2_{I,\Lambda}$, that equals (4.19), has an expansion in terms of the fields that is a polynomial in creation and annihilation operators of degree $2(\tilde{n}-1)$. Each monomial in this expansion, expressed in terms of creation and annihilation operators, has an l^2 kernel. As a consequence, there is a constant $M_3 = M_3(j, V)$, such that the purely bosonic interaction term $Q_{I,\Lambda}(V)^2$

satisfies the upper bound, $Q_{I,\Lambda}(V)^2 \leq M_3 \,(N+I)^{\tilde{n}-1}$. This estimate is a standard property of monomials in creation and annihilation operators with l^2-kernels; in the constructive quantum field theory literature this estimate is known as an N_τ-bound, and the contribution to the constant M_3 from each monomial is the l^2 norm of the corresponding kernel, see [3]. Because the twisting angle is fixed and lies in the interval $0 < \phi \leq \pi$, there is a constant $M_5 = M_5(\phi)$ such that the commuting operators N and H_0 satisfy $N \leq M_5 H_0$. Thus with a new choice of the constant M_3 (and suppressing the dependence on ϕ, which is fixed) we obtain the bounds (5.35) with $\alpha = 1$. The interpolation inequalities with $0 \leq \alpha \leq 1$ then follow from the Cauchy representation for the fractional powers of the resolvents, see Chapter V, Remark 3.50 of [16].

(ii) The bound

$$\pm\{Q_0, Q_{I,\Lambda}\} \leq Q_0^2 + Q_{I,\Lambda}^2 = H_0 + P + Q_{I,\Lambda}^2, \tag{5.38}$$

leads to the desired estimate with a new constant M_3. Use the elementary bound $P \leq H_0$ to estimate P, and use the bound (5.35) with $\alpha = 1$ to estimate $Q_{I,\Lambda}^2$.

(iii) The bound (5.5) with $1 \leq M_1$ and $I \leq H_\Lambda(\lambda V) + M_2$ ensures that

$$\lambda^2 \, Q_{I,\Lambda}(V)^2 \leq M_1 \,(H_\Lambda(\lambda V) + M_2)\,. \tag{5.39}$$

As a consequence, the domain of the operator $Q_{I,\Lambda}(V) = Q_{I,\Lambda}(V)$ contains all vectors in the domain of $(H_\Lambda(\lambda V) + M_2)^{1/2}$, for any $\lambda > 0$. It follows that we have an interpolation inequality: for any $\alpha \in [0, 1]$,

$$\lambda^{2(1-\alpha)} \, |Q_{I,\Lambda}(V)|^{2(1-\alpha)} \leq M_1^{(1-\alpha)} \,(H_\Lambda(\lambda V) + M_2)^{1-\alpha}\,. \tag{5.40}$$

The operator form of this inequality is

$$\left\| |Q_{I,\Lambda}(V)|^{1-\alpha} \,(H_\Lambda(\lambda V) + M_2)^{-(1-\alpha)/2} \right\| \leq M_1^{(1-\alpha)/2} \lambda^{-1+\alpha}. \tag{5.41}$$

Using part (i) of the lemma, we also have the operator interpolation inequality,

$$\left\| |Q_{I,\Lambda}(V)|^{\alpha} \,(N+I)^{-\alpha(\tilde{n}-1)/2} \right\| \leq M_1^{\alpha/2}. \tag{5.42}$$

Note that the bound (5.42) does not involve λ. Combining (5.41) and (5.42), and the self-adjointness of $Q_{I,\Lambda}$ and $H_\Lambda(\lambda V)$, we have

$$\begin{aligned} &\left\| (H_\Lambda(\lambda V) + M_2)^{-(1-\alpha)/2} \; Q_{I,\Lambda}(V) \; \big(H_\Lambda(\lambda' V) + M_2\big)^{-\alpha(\tilde{n}-1)/2} \right\| \\ &\quad \leq \left\| (H_\Lambda(\lambda V) + M_2)^{-(1-\alpha)/2} \; Q_{I,\Lambda}(V) \,(N+I)^{-\alpha(\tilde{n}-1)/2} \right\| \\ &\qquad \times \left\| (N+I)^{\alpha(\tilde{n}-1)/2} \; \big(H_\Lambda(\lambda' V) + M_2\big)^{-\alpha(\tilde{n}-1)/2} \right\| \\ &\quad \leq M_1^{(1-\alpha)/2} \lambda^{-1+\alpha}. \end{aligned} \tag{5.43}$$

We obtain the interpolation bound on

$$\left\| (N+I)^{\alpha(\tilde{n}-1)/2} \; \big(H_\Lambda(\lambda' V) + M_2\big)^{-\alpha(\tilde{n}-1)/2} \right\| \leq M_1^{\alpha(\tilde{n}-1)/2},$$

using (5.1), as long as $\alpha(\tilde{n}-1) \leq 1$, which we assume. This completes the proof of the lemma. □

Proof of Proposition 5.5. Expand $F_\Lambda^\beta(\lambda, \lambda', s)$ according to the definition (5.31). Write the first term $R(s\beta)\, Q_\Lambda(\lambda V)\, Q_{I,\Lambda}(V)\, R'((1-s)\beta)$ term in $-F(\lambda, \lambda', s)$ as the following product of four bounded operators separted in braces,

$$\begin{aligned} R(s\beta)\, Q_\Lambda(\lambda V)\, Q_{I,\Lambda}(V)\, R'((1-s)\beta) & \\ = \{R(s\beta/4)\}\, \{R(s\beta/4)\, Q_\Lambda(\lambda V)\, R(s\beta/4)\} & \\ \times \left\{R(s\beta/4)\, Q_{I,\Lambda}(V)\, R'(3(1-s)\beta/4)\right\} \left\{R'((1-s)\beta/4)\right\}. & \end{aligned} \tag{5.44}$$

Apply Hölder's inequality to bound the trace norm of this product of four terms, using the exponents $\frac{1}{s}, \infty, \infty, \frac{1}{1-s}$. Then

$$\begin{aligned} \left\| R(s\beta)\, Q_\Lambda(\lambda V)\, Q_{I,\Lambda}(V)\, R'((1-s)\beta) \right\|_1 & \\ \le \|R(s\beta/4)\|_{1/s}\, \|R(s\beta/4)\, Q_\Lambda(\lambda V)\, R(s\beta/4)\| & \\ \times \left\| R(s\beta/4)\, Q_{I,\Lambda}(V)\, R'(3(1-s)\beta/4) \right\| \, \left\| R'((1-s)\beta/4) \right\|_{1/(1-s)}. & \end{aligned} \tag{5.45}$$

Bound the first and last factors on the right of (5.45) using the uniform estimate (5.3). Thus

$$\|R(s\beta/4)\|_{1/s}\ \left\| R'((1-s)\beta/4) \right\|_{1/(1-s)} \le M(\beta/4). \tag{5.46}$$

Use the spectral theorem to bound the second factor on the right of (5.45) uniformly in λ, by

$$\|R(s\beta/2)\, Q_\Lambda(\lambda V)\, R(s\beta/4)\| \le O(1)\, s^{-1/2}, \tag{5.47}$$

where the constant in $O(1)$ depends on β and Λ , but not on λ. Bound the third factor in (5.45) by

$$\begin{aligned} \left\| R(s\beta/4)\, Q_{I,\Lambda}(V)\, R'((1-s)\beta/2) \right\| & \\ \le \left\| R(s\beta/4)\ (H_\Lambda(\lambda V) + M_2)^{(1-\alpha)/2} \right\| & \\ \times \left\| (H_\Lambda(\lambda V) + M_2)^{-(1-\alpha)/2}\, Q_{I,\Lambda}(V) \left(H_\Lambda(\lambda' V)\right)^{-\alpha(\tilde{n}-1)/2} \right\| & \\ \times \left\| \left(H_\Lambda(\lambda' V)\right)^{\alpha(\tilde{n}-1)/2}\, R'((1-s)\beta/2) \right\|. & \end{aligned} \tag{5.48}$$

The first factor on the right of (5.48) is $O(1)\, s^{-(1-\alpha)/2}$, again with the constant in $O(1)$ depending on β and Λ , but not on λ. From Lemma 5.6 we infer that the second factor in (5.48) is $O(\lambda^{-1+\alpha})$, with the same proviso about $O(1)$. Finally we estimate the third factor in (5.48) by $O(1)(1-s)^{-\alpha(\tilde{n}-1)/2}$, with $O(1)$ depending on Λ and β. These three bounds yield

$$\left\| R(s\beta/4)\, Q_{I,\Lambda}(V)\, R'((1-s)\beta/2) \right\| \le O(1)\, \lambda^{-1+\alpha}\, s^{-(1-\alpha)/2}\, (1-s)^{-\alpha(\tilde{n}-1)/2}. \tag{5.49}$$

We combine the estimates (5.46), (5.47), and (5.49) to obtain

$$\left\| R(s\beta)\, Q_\Lambda(\lambda V)\, Q_{I,\Lambda}(V)\, R'((1-s)\beta) \right\|_1 \leq O(1)\lambda^{-1+\alpha}\, s^{-1+\alpha/2}\, (1-s)^{-\alpha(\tilde{n}-1)/2}, \tag{5.50}$$

which is the first term in the bound (5.34).

In order to bound the second term $R(s\beta)\, Q_{I,\Lambda}(V)\, Q_\Lambda(\lambda' V)\, R'((1-s)\beta)$ in $-F(\lambda, \lambda', s)$, repeat this procedure, but use the adjoint bounds. This yields the estimate

$$\left\| R(s\beta)\, Q_{I,\Lambda}(V)\, Q_\Lambda(\lambda' V)\, R'((1-s)\beta) \right\|_1 \leq O(1)(\lambda')^{-1+\alpha}\, s^{-\alpha(\tilde{n}-1)/2}(1-s)^{-1+\alpha/2}. \tag{5.51}$$

Adding (5.50) and (5.51) completes the proof of the desired estimate (5.34).

We now establish statement (ii). Let $0 < s < s' < 1$. Using the bound (5.1), we infer that there is a constant M such that, for $\lambda \in [0, 1]$, the heat kernel $e^{-s\beta H_\Lambda(\lambda V)}$ is bounded in norm by $M^{s\beta}$. Therefore $\left\| e^{-s\beta H_\Lambda(\lambda V)} - e^{-s'\beta H_\Lambda(\lambda V)} \right\| \leq 2M^\beta$. We can also bound the difference

$$e^{-s\beta H_\Lambda(\lambda V)} - e^{-s'\beta H_\Lambda(\lambda V)} = \left(I - e^{-(s'-s)\beta H_\Lambda(\lambda V)} \right) e^{-s\beta H_\Lambda(\lambda V)}, \tag{5.52}$$

using the fundamental theorem of calculus, giving

$$\left\| e^{-s\beta H_\Lambda(\lambda V)} - e^{-s'\beta H_\Lambda(\lambda V)} \right\| \leq M^\beta\, |s'-s|/s.$$

Combining these two bounds on the difference, we infer that there is a new constant $M > 1$ such that for any $0 < \epsilon' \leq 1$,

$$\left\| e^{-s\beta H_\Lambda(\lambda V)} - e^{-s'\beta H_\Lambda(\lambda V)} \right\| \leq M^\beta\, \left(|s'-s|/s \right)^{\epsilon'}. \tag{5.53}$$

The same bounds hold with $H_\Lambda(\lambda V)$ replaced by $H_\Lambda(\lambda' V)$. To simplify notation, let us denote $H = H_\Lambda(\lambda V)$, $H' = H_\Lambda(\lambda' V)$, $Q = Q_\Lambda(\lambda V)$, $Q' = Q_\Lambda(\lambda' V)$, and $Q_I = Q_{I,\Lambda}(V)$. Now write the difference

$$\begin{aligned} &F^\beta_\Lambda(\lambda, \lambda', s) - F^\beta_\Lambda(\lambda, \lambda', s') \\ &= \beta e^{-s\beta H} \left(Q\, Q_I + Q_I\, Q' \right) e^{-(1-s)\beta H'} - \beta e^{-s'\beta H} \left(Q\, Q_I + Q_I\, Q' \right) e^{-(1-s')\beta H'} \\ &= \beta \left(I - e^{-(s'-s)\beta H} \right) e^{-s\beta H/2}\, F^{\beta/2}(\lambda, \lambda', s)\, e^{-(1-s)\beta H'/2} \\ &\quad + \beta e^{-s'\beta H/2}\, F^{\beta/2}(\lambda, \lambda', s')\, e^{-(1-s')\beta H'/2} \left(e^{-(s'-s)\beta H'} - I \right). \end{aligned} \tag{5.54}$$

From (5.53) and Hölder's inequality, we obtain for any $0 < \epsilon' \leq 1$,

$$\begin{aligned} &\left\| F^\beta_\Lambda(\lambda, \lambda', s) - F^\beta_\Lambda(\lambda, \lambda', s') \right\|_1 \\ &\leq \beta M^\beta |s'-s|^{\epsilon'} \left(s^{-\epsilon'} \left\| F^{\beta/2}(\lambda, \lambda', s) \right\|_1 + (1-s')^{-\epsilon'} \left\| F^{\beta/2}(\lambda, \lambda', s') \right\|_1 \right). \end{aligned} \tag{5.55}$$

Taking the bound (5.34) into account, we conclude in the case $0 < s < s' < 1$ that the map $s \mapsto F^\beta_\Lambda(\lambda, \lambda', s)$ is Hölder continuous in trace norm with an exponent ϵ'. A similar bound holds if $0 < s' < s < 1$, but with s and s' interchanged, completing the proof of the proposition. □

Corollary 5.7. *We have the following.*

(i) *Let $\eta > 0$. For any bounded operator A,*

$$\int_{\eta}^{1-\eta} \mathrm{Tr}_{\mathcal{H}} \left(A\, F(\lambda, \lambda', s)\right) ds = \mathrm{Tr}_{\mathcal{H}} \left(A \int_{\eta}^{1-\eta} F(\lambda, \lambda', s) ds\right). \tag{5.56}$$

(ii) *Let $\eta > 0$. The operators $\int_{\eta}^{1-\eta} F_{\Lambda}^{\beta}(\lambda, \lambda', s) ds$ converge in trace-norm as $\eta \to 0$, defining $\int_{0}^{1} F_{\Lambda}^{\beta}(\lambda, \lambda', s) ds$. Thus for any bounded A,*

$$\int_{0}^{1} \mathrm{Tr}_{\mathcal{H}} \left(A\, F(\lambda, \lambda', s)\right) ds = \mathrm{Tr}_{\mathcal{H}} \left(A \int_{0}^{1} F(\lambda, \lambda', s) ds\right). \tag{5.57}$$

(iii) *For $A = I$, this limit equals the difference quotient,*

$$\lim_{\eta \to 0} \left\| \Delta^{\beta}(\lambda, \lambda') - \int_{\eta}^{1-\eta} F_{\Lambda}^{\beta}(\lambda, \lambda', s) ds \right\|_{1} = 0, \tag{5.58}$$

and

$$\Delta^{\beta}(\lambda, \lambda') = \int_{0}^{1} F_{\Lambda}^{\beta}(\lambda, \lambda', s) ds. \tag{5.59}$$

(iv) *For any bounded operator A,*

$$\mathrm{Tr}_{\mathcal{H}} \left(A\, \Delta^{\beta}(\lambda, \lambda')\right) = \int_{0}^{1} \mathrm{Tr}_{\mathcal{H}} \left(A\, F_{\Lambda}^{\beta}(\lambda, \lambda', s)\right) ds, \tag{5.60}$$

yielding the estimate

$$\left| \mathrm{Tr}_{\mathcal{H}} \left(A \Delta^{\beta}(\lambda, \lambda')\right) \right| \leq O(\lambda_{\min}^{-1+\alpha}) \|A\|. \tag{5.61}$$

Proof. Statement (i) of the corollary follows from the continuity of $F_{\Lambda}^{\beta}(\lambda, \lambda', s)$ in s, namely Proposition 5.5.ii. Statement (ii) of the corollary is a consequence of the estimate of Proposition 5.5.i. We now verify (iii). Consider the domain $\mathcal{D}_s \times \mathcal{D}'_{1-s} = e^{-sH_{\Lambda}(\lambda V)}\mathcal{H} \times e^{-(1-s)H_{\Lambda}(\lambda' V)}\mathcal{H}$. Both $H_{\Lambda}(\lambda V)$ and $H_{\Lambda}(\lambda' V)$ are sesqui-linear forms on this domain. Furthermore, from Proposition 5.3 we infer that both $H_{\Lambda}(\lambda V) = Q_{\Lambda}(\lambda V)^2 - P$ and $H_{\Lambda}(\lambda' V) = Q_{\Lambda}(\lambda' V)^2 - P$ on this domain. Therefore, we have the identity of forms,

$$\begin{aligned} H_{\Lambda}(\lambda V) - H_{\Lambda}(\lambda' V) &= Q_{\Lambda}(\lambda V)^2 - Q_{\Lambda}(\lambda' V)^2 \\ &= Q_{\Lambda}(\lambda V) \left(Q_{\Lambda}(\lambda V) - Q_{\Lambda}(\lambda' V)\right) \\ &\quad + \left(Q_{\Lambda}(\lambda V) - Q_{\Lambda}(\lambda' V)\right) Q_{\Lambda}(\lambda' V) \\ &= \left(\lambda - \lambda'\right) \left(Q_{\Lambda}(\lambda V) Q_{I,\Lambda}(V) + Q_{I,\Lambda}(V) Q_{\Lambda}(\lambda' V)\right) \end{aligned} \tag{5.62}$$

on $\mathcal{D}_s \times \mathcal{D}'_{1-s}$. Consequently, on $\mathcal{H} \times \mathcal{H}$,

$$e^{-sH_{\Lambda}(\lambda V)} \left(\frac{H_{\Lambda}(\lambda V) - H_{\Lambda}(\lambda' V)}{\lambda - \lambda'}\right) e^{-(1-s)H_{\Lambda}(\lambda' V)} = -F_{\Lambda}^{\beta}(\lambda, \lambda', s). \tag{5.63}$$

Part (ii) of the corollary asserts that this expression has an integral over $s \in [\eta, 1-\eta]$, that converges in trace norm as $\eta \to 0$. Therefore

$$\int_0^1 e^{-sH_\Lambda(\lambda V)} \left(\frac{H_\Lambda(\lambda' V) - H_\Lambda(\lambda V)}{\lambda - \lambda'} \right) e^{-(1-s)H_\Lambda(\lambda' V)} ds = \int_0^1 F_\Lambda^\beta(\lambda, \lambda', s) ds. \tag{5.64}$$

But the left side of this identity is the difference quotient $\Delta^\beta(\lambda, \lambda')$, so we have identified the $\eta \to 0$ limit. Finally, the same argument proves that

$$\lim_{\eta \to 0} \left\| A\, \Delta^\beta(\lambda, \lambda') - \int_\eta^{1-\eta} A\, F_\Lambda^\beta(\lambda, \lambda', s) ds \right\|_1 = 0, \tag{5.65}$$

and the bounds of statement (iv) then follow from integrating the estimate of Proposition 5.5.i. □

6. Estimates on Traces

In this section, we estimate certain partition functions. Their proof involves further cancellations, that are not captured by the estimates studied in the previous section. The proofs here use the estimates on operators from the previous section, both to justify the existence of the objects studied here, as well as to estimate the quantities that arise after exhibiting cancellations in the trace that defines the partition functions.

6.1. Differentiability for $\lambda > 0$. In this section we establish differentiability of $\mathfrak{Z}_\Lambda^{\lambda V}$ as a function of λ. Choose the bounded operator A in Corollary 5.7.iv to be $A = \Gamma e^{-i\theta J - i\sigma P}$. Then the corollary yields a representation for the difference quotient

$$\begin{aligned} \delta_\Lambda(\lambda, \lambda') = \frac{\mathfrak{Z}_\Lambda^{\lambda V} - \mathfrak{Z}_\Lambda^{\lambda' V}}{\lambda - \lambda'} &= \lim_{\eta \to 0} \int_\eta^{1-\eta} \mathrm{Tr}_{\mathcal{H}} \left(A\, F_\Lambda^\beta(\lambda, \lambda', s) \right) ds \\ &= \int_0^1 \mathrm{Tr}_{\mathcal{H}} \left(A\, F_\Lambda^\beta(\lambda, \lambda', s) \right) ds. \end{aligned} \tag{6.1}$$

Furthermore, the putative derivative of $\mathfrak{Z}_\Lambda^{\lambda V}$ also has an integral representation, namely

$$\delta_\Lambda(\lambda, \lambda) = \lim_{\eta \to 0} \int_\eta^{1-\eta} \mathrm{Tr}_{\mathcal{H}} \left(A\, F_\Lambda^\beta(\lambda, \lambda, s) \right) ds = \int_0^1 \mathrm{Tr}_{\mathcal{H}} \left(A\, F_\Lambda^\beta(\lambda, \lambda, s) \right) ds. \tag{6.2}$$

Although both representations (6.1) and (6.2) are well defined, we have not established that $\delta_\Lambda(\lambda, \lambda')$ has a limit as $\lambda' \to \lambda$, nor if this limit exists whether it equals $\delta_\Lambda(\lambda, \lambda)$. In this section we find the consequence of the smoothing provided by the specific operator A in the partition function, This allows us to prove differentiability of the partition function, and actually its vanishing.

Theorem 6.1. *Under the conditions of Theorem 2.1, the map $\lambda \mapsto \mathfrak{Z}_\Lambda^{\lambda V}$ is a differentiable function of λ for all $\lambda \in (0, 1]$. In fact, the derivative vanishes, $\frac{\partial}{\partial \lambda} \mathfrak{Z}_\Lambda^{\lambda V} = \delta_\Lambda(\lambda, \lambda) = 0$.*

Proof. The bounds in the previous section show that $\delta_\Lambda(\lambda, \lambda')$ is bounded. To establish the theorem, we show that the difference quotient (6.1) actually converges to zero,

$$\lim_{\lambda' \to \lambda} \delta_\Lambda(\lambda, \lambda') = 0, \tag{6.3}$$

for $\lambda > 0$. A similar argument shows that $\delta(\lambda, \lambda) = 0$.

We claim that for each fixed $\lambda \in (0, 1]$, there exists a positive, constant $M = M(\beta, \Lambda, \lambda, V)$, not depending on λ', such that

$$\left| \operatorname{Tr}_{\mathcal{H}} \left(A F^\beta(\lambda, \lambda', s) \right) \right| \le M \, |\lambda - \lambda'| \, s^{-1/2} \, (1-s)^{-1/2}, \tag{6.4}$$

whenever $\lambda' \in (0, 1]$ lies in the neighborhood of λ defined by $\mathcal{B}_\lambda = \{\lambda' : |\lambda' - \lambda| \le \frac{1}{2}\lambda\}$. Let us assume this bound. As a consequence,

$$\begin{aligned} \left| \int_0^1 \operatorname{Tr}_{\mathcal{H}} \left(A F^\beta(\lambda, \lambda', s) \right) ds \right| &= \lim_{\eta \to 0} \left| \int_\eta^{1-\eta} \operatorname{Tr}_{\mathcal{H}} \left(A F^\beta(\lambda, \lambda', s) \right) ds \right| \\ &\le \pi \, M \, |\lambda - \lambda'|, \end{aligned} \tag{6.5}$$

for $\lambda, \lambda' \in (0, 1]$ and $\lambda' \in \mathcal{B}_\lambda$. Thus according to the representation (6.1),

$$\left| \delta_\Lambda(\lambda, \lambda') \right| \le \pi M \, |\lambda - \lambda'|, \tag{6.6}$$

for $\lambda, \lambda' \in (0, 1]$ and $\lambda' \in \mathcal{B}_\lambda$, and the derivative of $\mathfrak{Z}_\Lambda^{\lambda V}$ vanishes as claimed.

Thus we have reduced the proof of the theorem to the proof of (6.4), which we now establish. We use the notation in the proof of Proposition 5.5. Write the density $\operatorname{Tr}_{\mathcal{H}} \left(A \, F_\Lambda^\beta(\lambda, \lambda', s) \right)$ for the difference quotient as

$$\begin{aligned} \operatorname{Tr}_{\mathcal{H}} \left(A \, F_\Lambda^\beta(\lambda, \lambda', s) \right) &= \beta \operatorname{Tr}_{\mathcal{H}} \left(A \, R(s\beta) \, Q \, Q_I \, R'((1-s)\beta) \right) \\ &\quad + \beta \operatorname{Tr}_{\mathcal{H}} \left(A \, R(s\beta) \, Q_I \, Q' \, R'((1-s)\beta) \right). \end{aligned} \tag{6.7}$$

The operator A commutes with R and with R' and it anticommutes with Q, Q', and Q_I. Also, we have seen that $R(s\beta) \, Q_I$ and $Q \, R'((1-s)\beta)$ are both trace class. Therefore using cyclicity of the trace,

$$\begin{aligned} \operatorname{Tr}_{\mathcal{H}} \left(A \, F_\Lambda^\beta(\lambda, \lambda', s) \right) &= -\beta \operatorname{Tr}_{\mathcal{H}} \left(A \, Q_I \, R'((1-s)\beta) \, Q \, R(s\beta) \right) \\ &\quad + \beta \operatorname{Tr}_{\mathcal{H}} \left(A \, R(s\beta) \, Q_I \, Q' \, R'((1-s)\beta) \right). \end{aligned} \tag{6.8}$$

The bound (5.5) assures that the range of R is in the domain of both Q_0 and Q_I, and hence in the domain of both Q and Q'. Thus in the first term, we can write $Q = Q' + \left(Q - Q' \right) = Q' + \left(\lambda - \lambda' \right) Q_I$, to yield

$$\begin{aligned} \operatorname{Tr}_{\mathcal{H}} \left(A \, F_\Lambda^\beta(\lambda, \lambda', s) \right) &= -\beta \operatorname{Tr}_{\mathcal{H}} \left(A \, R(s\beta) \, Q_I \, R'((1-s)\beta) \, Q' \right) \\ &\quad + \beta \left(\lambda - \lambda' \right) \operatorname{Tr}_{\mathcal{H}} \left(A \, Q_I \, R'((1-s)\beta) \, Q_I \, R(s\beta) \right) \\ &\quad + \beta \operatorname{Tr}_{\mathcal{H}} \left(A \, R(s\beta) \, Q_I \, Q' \, R'((1-s)\beta) \right) \\ &= -\beta \operatorname{Tr}_{\mathcal{H}} \left(A \, R(s\beta) \, Q_I \, Q' \, R'((1-s)\beta) \right) \\ &\quad + \beta \left(\lambda - \lambda' \right) \operatorname{Tr}_{\mathcal{H}} \left(A \, Q_I \, R'((1-s)\beta) \, Q_I \, R(s\beta) \right) \\ &\quad + \beta \operatorname{Tr}_{\mathcal{H}} \left(A \, R(s\beta) \, Q_I \, Q' \, R'((1-s)\beta) \right) \\ &= \beta \left(\lambda - \lambda' \right) \operatorname{Tr}_{\mathcal{H}} \left(A \, Q_I \, R'((1-s)\beta) \, Q_I \, R(s\beta) \right) \\ &= \beta \left(\lambda - \lambda' \right) \operatorname{Tr}_{\mathcal{H}} \left(A \, R(s\beta/2) \, Q_I \, R'((1-s)\beta) \, Q_I \, R(s\beta/2) \right). \end{aligned} \tag{6.9}$$

We estimate (6.9) using Hölder's inequality, obtaining

$$\begin{aligned} &\left| \operatorname{Tr}_{\mathcal{H}} \left(A\, F_{\Lambda}^{\beta}(\lambda, \lambda', s) \right) \right| \\ &\qquad \le \beta\, |\lambda - \lambda'|\, \|A\|\, \|R(s\beta/4)\|_s \left\| R(s\beta/4)\, Q_I\, R'((1-s)\beta/4) \right\| \\ &\qquad\quad \times \left\| R'((1-s)\beta/4) \right\|_{1-s} \left\| R'((1-s)\beta/2)\, Q_I\, R(s\beta/2) \right\| \\ &\qquad \le \beta\, |\lambda - \lambda'|\, M(\beta/4) \left\| R(s\beta/4)\, Q_I\, R'((1-s)\beta/4) \right\|^2 . \end{aligned} \tag{6.10}$$

The constant $M(\beta/4)$ in the last term is the constant in (5.3), and the bound on Q_I involves the self- adjointness of Q_I, R, and R'. From (5.5) we infer that with a new constant $M_4 = M_4(\beta, \Lambda, V)$,

$$\left| \operatorname{Tr}_{\mathcal{H}} \left(A\, F_{\Lambda}^{\beta}(\lambda, \lambda', s) \right) \right| \le \beta\, |\lambda - \lambda'|\, M_4\, \lambda^{-1}\, \lambda'^{-1}\, s^{-1/2}\, (1-s)^{-1/2}. \tag{6.11}$$

On the set $\mathcal{B}_\lambda$, we have $\lambda' = \lambda + (\lambda' - \lambda) \ge \frac{1}{2}\lambda$. Thus taking $M(\beta, \Lambda, \lambda, V) = 2\beta\, M_4(\beta, \Lambda, V)\lambda^{-2}$, we establish (6.4), and complete the proof of the theorem. □

6.2. Hölder Continuity at $\lambda = 0$. In Theorem 6.1, we found that the partition function $\mathfrak{Z}_\Lambda^{\lambda V}$ is a constant function of λ for all $\lambda \in (0, 1]$. At the $\lambda = 0$ endpoint of the interval, $H_\Lambda(\lambda V) = H_0$. If both $0 < \phi \le \pi$ and $0 < \beta$, then the heat kernel $e^{-\beta H_0}$ is trace class, and the partition function $\mathfrak{Z}^0 = \mathfrak{Z}_\Lambda^0$ is well defined. However, $\mathfrak{Z}_\Lambda^{\lambda V}$ might have a jump discontinuity at $\lambda = 0$, so it may not be the case that $\mathfrak{Z}_\Lambda^{\lambda V} = \mathfrak{Z}^0$. It is important to demonstrate the continuity of $\mathfrak{Z}_\Lambda^{\lambda V}$, and we do so by establishing Hölder continuity at $\lambda = 0$ with an exponent depending on the degree $\tilde{n} = \tilde{n}(V)$ of the polynomial potential V.

Theorem 6.2. *Assume the hypotheses of Theorem 2.1. Let $0 \le \alpha < 2/(\tilde{n} - 1)$. Then there exists a constant $M = M(\alpha, \beta, \Lambda, V)$ such that the partition function $\mathfrak{Z}_\Lambda^{\lambda V}$ satisfies*

$$\left| \mathfrak{Z}_\Lambda^{\lambda V} - \mathfrak{Z}^0 \right| \le M\, \lambda^\alpha, \quad \textit{for all } 0 < \lambda \le 1. \tag{6.12}$$

Corollary 6.3. *Under the hypotheses of Theorem 2.1, the functions $\mathfrak{Z}_\Lambda^{\lambda V}$ are independent of Λ and of λ, and*

$$\mathfrak{Z}_\Lambda^{\lambda V}(\tau, \theta, \phi) = \mathfrak{Z}^0(\tau, \theta, \phi), \quad \textit{for all } \lambda \in \mathbb{C}. \tag{6.13}$$

Proof. The corollary for $\lambda \in [0, 1]$ is an immediate consequence of the theorem. Substituting γV for V, with $\gamma \in \mathbb{C}$, we also have an allowed potential, and also $\mathfrak{Z}_\Lambda^{(\lambda\gamma)V} = \mathfrak{Z}_\Lambda^{\lambda(\gamma V)} = \mathfrak{Z}^0$. So the identity $\mathfrak{Z}_\Lambda^{\lambda V}(\tau, \theta, \phi) = \mathfrak{Z}^0(\tau, \theta, \phi)$ extends to all $\lambda \in \mathbb{C}$.

The first step in the proof of the theorem is to establish a representation for the difference $\mathfrak{Z}^0 - \mathfrak{Z}^{\lambda V}$, that is similar to the representation in the previous section for the difference quotient (5.60), except that it is convergent at the $\lambda = 0$ endpoint of the interval.

Lemma 6.4. *There are constants* $M_1 = M_1(j, V) < \infty$ *and* $M_2 = M_2(j, V) < \infty$ *such that*

$$I \leq H_\Lambda(\lambda V) + M_2 \leq M_1 \ (H_0 + I)^{\tilde{n}-1}\,, \tag{6.14}$$

and for all $0 \leq \alpha \leq 1$,

$$\left\| (H_\Lambda + M_2)^{\alpha/2} \, (H_0 + I)^{-\alpha(\tilde{n}-1)/2} \right\| \leq M_1^{\alpha/2}. \tag{6.15}$$

Proof. Write $H_\Lambda(\lambda V) = H_0 + \lambda^2 \, Q_{I,\Lambda}(V)^2 + \lambda \left(Y_\Lambda + Y_\Lambda^*\right)$, where $Y_\Lambda + Y_\Lambda^* = \{Q_0, Q_{I,\Lambda}\}$. Since $\tilde{n} \geq 2$, the upper bound (6.14) holds trivially for $\lambda = 0$. The bound of Lemma 5.6.i ensures that $Q_{I,\Lambda}^2 \leq M_3(H_0 + I)^{\tilde{n}-1}$. Finally, as a consequence of Lemma 5.6.ii, the term $Y_\Lambda + Y_\Lambda^*$ is bounded from above by $M_3(H_0 + I)^{\tilde{n}-1}$. Taken together, these bounds establish (6.14). We choose M_2 sufficiently large so that $I \leq H_\Lambda(\lambda V) + M_2$. The lemma then follows from the interpolation inequality $(H_\Lambda + M_2)^\alpha \leq M_1^\alpha (H_0 + I)^{\alpha(\tilde{n}-1)}$, valid for $0 \leq \alpha \leq 1$.

For $s \in (0, 1)$, define the operator-valued function

$$f_\Lambda^\beta(\lambda, s) = e^{-s\beta H_\Lambda(\lambda V)} \, (H_0 - H_\Lambda(\lambda V)) \, e^{-(1-s)\beta H_0}, \quad \text{for} \;\; s \in (0, 1). \tag{6.16}$$

Lemma 6.5. *Under the hypotheses of the theorem, and for* $s \in (0, 1)$,

(i) *Both* $e^{-s\beta H_\Lambda(\lambda V)} \, H_0 \, e^{-(1-s)\beta H_0}$ *and* $e^{-s\beta H_\Lambda(\lambda V)} \, H_\Lambda(\lambda V) \, e^{-(1-s)\beta H_0}$ *are trace class.*

(ii) *There exists a constant* $M_6 = M_6(\beta, \Lambda, V)$*, such that the function* $f_\Lambda^\beta(\lambda, s)$ *has a trace-norm bounded by*

$$\left\| f_\Lambda^\beta(\lambda, s) \right\|_1 \leq M_6 \, s^{-1+1/2(\tilde{n}-1)} \, (1-s)^{-1/2}. \tag{6.17}$$

(iii) *The map* $s \mapsto f_\Lambda^\beta(\lambda, s)$ *is continuous in the trace-norm topology.*

(iv) *The integral of* f_Λ^β *exists, and for any bounded linear transformation* A,

$$\begin{aligned} \int_0^1 \mathrm{Tr}_{\mathcal{H}} \left(A \, f_\Lambda^\beta(\lambda, s) \right) ds &= \lim_{\eta \to 0} \int_\eta^{1-\eta} \mathrm{Tr}_{\mathcal{H}} \left(A \, f_\Lambda^\beta(\lambda, s) \right) ds \\ &= \mathrm{Tr}_{\mathcal{H}} \left(\lim_{\eta \to 0} \int_\eta^{1-\eta} A \, f_\Lambda^\beta(\lambda, s) ds \right) \\ &= \mathrm{Tr}_{\mathcal{H}} \left(\int_0^1 A \, f_\Lambda^\beta(\lambda, s) ds \right). \end{aligned} \tag{6.18}$$

(v) *The difference* $\mathfrak{Z}_\Lambda^{\lambda V} - \mathfrak{Z}^0$ *has the representation,*

$$\mathfrak{Z}_\Lambda^{\lambda V} - \mathfrak{Z}^0 = \beta \int_0^1 \mathrm{Tr}_{\mathcal{H}} \left(A \, f_\Lambda^\beta(\lambda, s) \right) ds, \tag{6.19}$$

where $A = \Gamma \, e^{-i\theta J - i\sigma P}$.

Proof. Write

$$\left\| e^{-s\beta H_\Lambda}\, H_0\, e^{-(1-s)\beta H_0} \right\|_1 \le \left\| e^{-s\beta H_\Lambda/2} \right\|_{s^{-1}} \left\| e^{-s\beta H_\Lambda/2}\, H_0\, e^{-(1-s)\beta H_0/2} \right\| \times \left\| e^{-(1-s)\beta H_0/2} \right\|_{(1-s)^{-1}}. \tag{6.20}$$

Hence using (5.3) and also (5.5), we conclude that there is a constant $M_6 = M_6(\beta, \Lambda, V)$ such that

$$\left\| e^{-s\beta H_\Lambda}\, H_0\, e^{-(1-s)\beta H_0} \right\|_1 \le M_6\, s^{-1/2}\, (1-s)^{-1/2}, \tag{6.21}$$

so $e^{-s\beta H_\Lambda}\, H_0\, e^{-(1-s)\beta H_0}$ is trace class.

With a possibly larger constant $M_6(\beta, \Lambda, V)$, we also have the bound

$$\begin{aligned} \left\| e^{-s\beta H_\Lambda}\, H_\Lambda\, e^{-(1-s)\beta H_0} \right\|_1 &\le \left\| e^{-s\beta H_\Lambda/2} \right\|_{s^{-1}} \left\| e^{-s\beta H_\Lambda/2}\, (H_\Lambda + M_2)^{1-1/2(\tilde{n}-1)} \right\| \\ &\quad \times \left\| (H_\Lambda + M_2)^{1/2(\tilde{n}-1)}\, (H_0 + I)^{-1/2} \right\| \\ &\quad \times \left\| (H_0 + I)^{1/2}\, e^{-(1-s)\beta H_0/2} \right\| \left\| e^{-\beta H_0/2} \right\|^{1-s} \\ &\le M_6\, s^{-1+1/2(\tilde{n}-1)}\, (1-s)^{-1/2}, \end{aligned} \tag{6.22}$$

where we use the bound of Lemma 6.4 to bound the third term of (6.22), as well as (5.3) to estimate the product of the first and last terms. This proves that $e^{-s\beta H_\Lambda}\, H_\Lambda\, e^{-(1-s)\beta H_0}$ is trace class. As $\tilde{n} \ge 2$, the two bounds (6.21) and (6.22) taken together yield the proof of (i–ii).

Let us use the notation $R(s) = e^{-s\beta H_\Lambda(\lambda V)}$ and $R_0(s) = e^{-s\beta H_0}$. In order to establish (iii), take $s < s'$ and consider the difference

$$\begin{aligned} &\left\| f_\Lambda^\beta(\lambda, s) - f_\Lambda^\beta(\lambda, s') \right\|_1 \\ &\le \left\| \big(R(s) - R(s')\big)\, (H_0 - H_\Lambda)\, R_0(1-s) \right\|_1 \\ &\quad + \left\| R(s')\, (H_0 - H_\Lambda)\, \big(R_0(1-s) - R_0(1-s')\big) \right\|_1 \\ &= \left\| \Big(\big(I - R(s'-s)\big)\, R(s/2)\Big)\, R(s/2)\, (H_0 - H_\Lambda)\, R_0(1-s) \right\|_1 \\ &\quad + \left\| R(s')\, (H_0 - H_\Lambda)\, R_0((1-s')/2)\, \Big(R_0((1-s')/2)\, \big(R_0(s'-s) - I\big)\Big) \right\|_1. \end{aligned} \tag{6.23}$$

We bound this using Hölder's inequality by

$$\begin{aligned} &\left\| f_\Lambda^\beta(\lambda, s) - f_\Lambda^\beta(\lambda, s') \right\|_1 \\ &\le \left\| \big(I - R(s'-s)\big)\, R(s/2) \right\| \left\| R(s/2)\, (H_0 - H_\Lambda)\, R_0((1-s)/2) \right\|_1 \\ &\quad \times \left\| R_0((1-s)/2) \right\| \\ &\quad + \left\| R(s'/2) \right\| \left\| R(s'/2)\, (H_0 - H_\Lambda)\, R_0((1-s')/2) \right\|_1 \\ &\quad \times \left\| R_0((1-s')/2)\, \big(R_0(s'-s) - I\big) \right\| \\ &\le \left\| \big(I - R(s'-s)\big)\, R(s/2) \right\| \left\| R_0((1-s)/2) \right\| \left\| f_\Lambda^{\beta/2}(\lambda, s) \right\|_1 \\ &\quad + \left\| \big(I - R_0(s'-s)\big)\, R_0((1-s')/2) \right\| \left\| R(s'/2) \right\| \left\| f_\Lambda^{\beta/2}(\lambda, s') \right\|_1. \end{aligned} \tag{6.24}$$

Use the bound (5.53), with $0 < \epsilon' < \frac{1}{2(\tilde{n}-1)}$, as well as Lemma 6.5.ii, to obtain with a new constant $M_6 = M_6(\beta, \Lambda, V)$,

$$\left\| f_\Lambda^\beta(\lambda, s) - f_\Lambda^\beta(\lambda, s') \right\|_1 \le M_6 \left| s' - s \right|^{\epsilon'} s^{-1+1/2(\tilde{n}-1)-\epsilon'} (1-s)^{-1/2} + M_6 \left| s' - s \right|^{\epsilon'} (s')^{-1+1/2(\tilde{n}-1)} (1-s')^{-1/2-\epsilon'}. \tag{6.25}$$

This establishes continuity. The proof of (iv) follows the proof of Corollary 5.7, and we omit the details. Taking $A = \Gamma e^{-i\theta J - i\sigma P}$, and observing that

$$\frac{\partial}{\partial s} \left(e^{-s\beta H_\Lambda(\lambda V)} e^{-(1-s)\beta H_0} \right) = f_\Lambda^\beta(\lambda, s)$$

yields (v). This completes the proof of the lemma. □

Lemma 6.6. *Assume the hypotheses of Theorem 2.1, take $A = \Gamma e^{-i\theta J - i\sigma P}$, and let $s \in (0, 1)$.*

(i) *We have the identity*

$$\mathrm{Tr}_{\mathcal{H}} \left(A\, f_\Lambda^\beta(\lambda, s) \right) = -\lambda^2 \mathrm{Tr}_{\mathcal{H}} \left(A\, e^{-(1-s)\beta H_0/2}\, Q_{I,\Lambda}(V)\, e^{-s\beta H_\Lambda(\lambda V)}\, Q_{I,\Lambda}(V)\, e^{-(1-s)\beta H_0/2} \right). \tag{6.26}$$

(ii) *There exists a constant $M_7 = M_7(\beta, \Lambda, V)$ such that for all $\alpha \in [0, 1/(\tilde{n}-1)]$,*

$$\left| \mathrm{Tr}_{\mathcal{H}} \left(A\, f_\Lambda^\beta(\lambda, s) \right) \right| \le M_7\, \lambda^{2\alpha}\, s^{-1+\alpha}\, (1-s)^{-\alpha(\tilde{n}-1)}. \tag{6.27}$$

Proof. Part (i) of the lemma is a consequence of the fact that both $e^{-s\beta H_\Lambda(\lambda V)}$ and $e^{-(1-s)H_0}$ are trace class. Furthermore, the bound $\pm P \le H_0$ along with Proposition 5.3 establishing a similar upper bound with H_Λ, shows that $e^{-s\beta H_\Lambda(\lambda V)}\, P\, e^{-(1-s)H_0}$ is trace class. We therefore rewrite $H_0 - H_\Lambda(\lambda V)$ in $f_\Lambda^\beta(\lambda, s)$ as

$$\begin{aligned} H_0 - H_\Lambda(\lambda V) &= H_0 + P - H_\Lambda(\lambda V) - P \\ &= Q_0^2 - Q_\Lambda(\lambda V)^2 = Q_\Lambda(\lambda V)\,(Q_0 - Q_\Lambda(\lambda V)) \\ &\quad + (Q_0 - Q_\Lambda(\lambda V))\, Q_0. \end{aligned}$$

Thus

$$\begin{aligned} f_\Lambda^\beta \lambda, s) &= e^{-s\beta H_\Lambda(\lambda V)}\, \left(Q_\Lambda(\lambda V)\,(Q_0 - Q_\Lambda(\lambda V)) + (Q_0 - Q_\Lambda(\lambda V))\, Q_0\right)\, e^{-(1-s)H_0} \\ &= -\lambda\, e^{-s\beta H_\Lambda(\lambda V)} \left(Q_\Lambda(\lambda V)\, Q_{I,\Lambda}(V) + Q_{I,\Lambda}(V)\, Q_0 \right) e^{-(1-s)H_0}. \end{aligned} \tag{6.28}$$

Furthermore, in the first term $Q_\Lambda(\lambda V)$ commutes with the heat kernel mollifier on the left, so the above methods show $e^{-s\beta H_\Lambda(\lambda V)}\, Q_\Lambda(\lambda V)\, Q_{I,\Lambda}(V)\, e^{-(1-s)H_0}$ is trace class. Similarly, Q_0 commutes with the mollifier on the right, so the second term is also trace class. Consider the first term. The operator $e^{-s\beta H_\Lambda(\lambda V)/2}\, Q_\Lambda(\lambda V)$ is bounded, the operator $e^{-s\beta H_\Lambda(\lambda V)/2}\, Q_{I,\Lambda}(V)\, e^{-(1-s)H_0}$ is trace class, and A anti-commutes with $e^{-s\beta H_\Lambda(\lambda V)/2}\, Q_\Lambda(\lambda V)$. Thus using cyclicity (5.9) one can write,

$$\begin{aligned}
& -\lambda\, \mathrm{Tr}_{\mathcal{H}}\left(Ae^{-s\beta H_\Lambda(\lambda V)}\, Q_\Lambda(\lambda V)\, Q_{I,\Lambda}(V)\, e^{-(1-s)H_0}\right) \\
&\quad = -\lambda\, \mathrm{Tr}_{\mathcal{H}}\left(Ae^{-s\beta H_\Lambda(\lambda V)/2}\, Q_\Lambda(\lambda V)\, e^{-s\beta H_\Lambda(\lambda V)/2}\, Q_{I,\Lambda}(V)\, e^{-(1-s)H_0}\right) \\
&\quad = \lambda\, \mathrm{Tr}_{\mathcal{H}}\left(A\, e^{-s\beta H_\Lambda(\lambda V)/2}\, Q_{I,\Lambda}(V)\, e^{-(1-s)H_0} e^{-s\beta H_\Lambda(\lambda V)/2}\, Q_\Lambda(\lambda V)\right) \\
&\quad = \lambda\, \mathrm{Tr}_{\mathcal{H}}\left(A\, e^{-s\beta H_\Lambda(\lambda V)/2}\, Q_{I,\Lambda}(V)\, e^{-(1-s)H_0}\, Q_\Lambda(\lambda V)\, e^{-s\beta H_\Lambda(\lambda V)/2}\right) \\
&\quad = \lambda\, \mathrm{Tr}_{\mathcal{H}}\left(A\, e^{-s\beta H_\Lambda(\lambda V)/2}\, Q_{I,\Lambda}(V)\, e^{-(1-s)H_0}\, \left(Q_0 + \lambda Q_{I,\Lambda}(V)\right)\, e^{-s\beta H_\Lambda(\lambda V)/2}\right) \\
&\quad = \lambda\, \mathrm{Tr}_{\mathcal{H}}\left(A\, e^{-s\beta H_\Lambda(\lambda V)/2}\, Q_{I,\Lambda}(V)\, Q_0\, e^{-(1-s)H_0}\, e^{-s\beta H_\Lambda(\lambda V)/2}\right) \\
&\qquad + \lambda^2\, \mathrm{Tr}_{\mathcal{H}}\left(A\, e^{-s\beta H_\Lambda(\lambda V)/2}\, Q_{I,\Lambda}(V)\, e^{-(1-s)H_0}\, Q_{I,\Lambda}(V)\, e^{-s\beta H_\Lambda(\lambda V)/2}\right) \\
&\quad = \lambda\, \mathrm{Tr}_{\mathcal{H}}\left(A\, e^{-s\beta H_\Lambda(\lambda V)}\, Q_{I,\Lambda}(V)\, Q_0\, e^{-(1-s)H_0}\right) \\
&\qquad + \lambda^2\, \mathrm{Tr}_{\mathcal{H}}\left(A\, e^{-s\beta H_\Lambda(\lambda V)/2}\, Q_{I,\Lambda}(V)\, e^{-(1-s)H_0}\, Q_{I,\Lambda}(V)\, e^{-s\beta H_\Lambda(\lambda V)/2}\right). \qquad (6.29)
\end{aligned}$$

On the other hand, since each term in (6.28) is trace class, we have

$$\begin{aligned}
\mathrm{Tr}_{\mathcal{H}}\left(A\, f_\Lambda^\beta(\lambda, s)\right) = & -\lambda\, \mathrm{Tr}_{\mathcal{H}}\left(A\, e^{-s\beta H_\Lambda(\lambda V)}\, Q_\Lambda(\lambda V)\, Q_{I,\Lambda}(V)\, e^{-(1-s)H_0}\right) \\
& -\lambda\, \mathrm{Tr}_{\mathcal{H}}\left(A\, e^{-s\beta H_\Lambda(\lambda V)}\, Q_{I,\Lambda}(V)\, Q_0\, e^{-(1-s)H_0}\right).
\end{aligned} \qquad (6.30)$$

Substituting (6.29) into (6.30), we obtain

$$\begin{aligned}
& \mathrm{Tr}_{\mathcal{H}}\left(A\, f_\Lambda^\beta(\lambda, s)\right) \\
&\quad = -\lambda^2\, \mathrm{Tr}_{\mathcal{H}}\left(A\, e^{-s\beta H_\Lambda(\lambda V)/2}\, Q_{I,\Lambda}(V)\, e^{-(1-s)H_0}\, Q_{I,\Lambda}(V)\, e^{-s\beta H_\Lambda(\lambda V)/2}\right), \qquad (6.31)
\end{aligned}$$

which proves (i).

In order to prove (ii), observe that a consequence of Lemma 5.6.iii, with $\alpha < (\tilde{n}-1)^{-1}$, is the following bound. There is a constant $M_8 = M_8(\beta, \Lambda, V)$, such that

$$\begin{aligned}
& \left\| e^{-s\beta H_\Lambda(\lambda V)/4}\, Q_{I,\Lambda}(V)\, e^{-(1-s)H_0/4} \right\| \\
&\qquad \leq \left\| e^{-s\beta H_\Lambda(\lambda V)/4}\, (H_\Lambda(\lambda V) + M_2)^{(1-\alpha)/2} \right\| \\
&\qquad\quad \times \left\| (H_\Lambda(\lambda V) + M_2)^{-(1-\alpha)/2}\, Q_{I,\Lambda}(V)\, (H_0 + I)^{-\alpha(\tilde{n}-1)/2} \right\| \\
&\qquad\quad \times \left\| (H_0 + I)^{\alpha(\tilde{n}-1)/2} e^{-(1-s)H_0/4} \right\| \\
&\qquad \leq M_8\, \lambda^{-1+\alpha}\, s^{-(1-\alpha)/2}\, (1-s)^{-\alpha(\tilde{n}-1)/2}. \qquad (6.32)
\end{aligned}$$

As a consequence of the representation (6.26), the fact that A is unitary, and using (5.8), we have

$$
\begin{aligned}
&\left| \mathrm{Tr}_{\mathcal{H}} \left(A\, f_\Lambda^\beta(\lambda, s) \right) \right| \\
&\quad \le \lambda^2 \left| \mathrm{Tr}_{\mathcal{H}} \left(A\, e^{-(1-s)\beta H_0/2}\, Q_{I,\Lambda}(V)\, e^{-s\beta H_\Lambda(\lambda V)}\, Q_{I,\Lambda}(V)\, e^{-(1-s)\beta H_0/2} \right) \right| \\
&\quad \le \lambda^2 \left\| e^{-(1-s)\beta H_0/2}\, Q_{I,\Lambda}(V)\, e^{-s\beta H_\Lambda(\lambda V)}\, Q_{I,\Lambda}(V)\, e^{-(1-s)\beta H_0/2} \right\|_1 \\
&\quad \le \lambda^2 \left\| e^{-(1-s)\beta H_0/4} \right\|_{1/(1-s)} \left\| e^{-(1-s)\beta H_0/4}\, Q_{I,\Lambda}(V)\, e^{-s\beta H_\Lambda(\lambda V)/4} \right\| \\
&\qquad \times \left\| e^{-s\beta H_\Lambda(\lambda V)/2} \right\|_{1/s} \left\| e^{-s\beta H_\Lambda(\lambda V)/4}\, Q_{I,\Lambda}(V)\, e^{-(1-s)\beta H_0/2} \right\| \qquad (6.33) \\
&\quad \le \lambda^2 \left\| e^{-\beta H_0/4} \right\|_1^{1-s} \left\| e^{-\beta H_\Lambda(\lambda V)/2} \right\|_1^s \left\| e^{-s\beta H_\Lambda(\lambda V)/4}\, Q_{I,\Lambda}(V)\, e^{-(1-s)\beta H_0/4} \right\|^2 .
\end{aligned}
$$

We have used Hölder's inequality with the exponents $(1-s)^{-1}$, ∞, s^{-1}, ∞, and the fact that $\|T\| = \|T^*\|$, as well as $\left\| e^{-(1-s)\beta H_0/4} \right\| \le 1$. We use the bound (5.3), along with (6.32), to complete the proof of (6.27). □

Proof of Theorem 6.2. Bound the difference $\left| \mathfrak{Z}_\Lambda^{\lambda V} - \mathfrak{Z}^0 \right|$ by using the representation of Lemma 6.5.v, and the bound of Lemma 6.6.ii. Integrating this bound, we obtain for any $\alpha \in (0, (\tilde{n}-1)^{-1})$,

$$
\begin{aligned}
\left| \mathfrak{Z}_\Lambda^{\lambda V} - \mathfrak{Z}^0 \right| &\le \beta \int_0^1 \left| \mathrm{Tr}_{\mathcal{H}} \left(A\, f_\Lambda^\beta(\lambda, s) \right) \right| ds \\
&\le \beta M_7\, \Gamma(\alpha)\, \Gamma(1-\alpha(\tilde{n}-1))\, \Gamma(1-\alpha(\tilde{n}-2))^{-1}\, \lambda^{2\alpha} .
\end{aligned} \qquad (6.34)
$$

The parameter 2α in the bound (6.34) becomes α in the bound (6.12). Thus we obtain Hölder continuity with any Hölder exponent strictly less than $2/(\tilde{n}-1)$, and the proof of the theorem is complete. □

Proof of Theorem 2.1. The bound (5.4), along with Proposition 5.1, ensures that the limit of partition functions $\lim_{j\to\infty} \mathfrak{Z}_\Lambda^{\lambda V}$ actually equals $\mathfrak{Z}^{\lambda V}$. There is no question about the existence or the numerical value of the limit: Theorem 6.1 ensures that the function $\mathfrak{Z}_\Lambda^{\lambda V}$ is constant in λ for $\lambda > 0$, and Theorem 6.2 ensures that $\mathfrak{Z}_\Lambda^{\lambda V}$ equals the same function at $\lambda = 0$. Since $\mathfrak{Z}^0$ is Λ -independent, therefore $\mathfrak{Z}_\Lambda^{\lambda V}$ is also Λ -independent. As a result, not only do the differentiability and continuity of $\mathfrak{Z}_\Lambda^{\lambda V}$ also hold for $\mathfrak{Z}^{\lambda V}$, but $\mathfrak{Z}^{\lambda V}$ is also λ-independent for $\lambda \in [0, 1]$. So we have established Theorem 2.1 and the first statement in Corollary 2.2.

7. Analyticity

In the previous section, we saw that $\mathfrak{Z}^V(\tau, \theta, \phi) = \mathfrak{Z}_\Lambda^V(\tau, \theta, \phi) = \mathfrak{Z}^0(\tau, \theta, \phi)$. In the next section we calculate $\mathfrak{Z}^0(\tau, \theta, \phi)$ and find that it is holomorphic for all $\tau \in \mathbb{H}$ and all $\theta \in \mathbb{C}$. Furthermore, it actually extends to a holomorphic function of ϕ. (There is an independent way to verify that $\mathfrak{Z}_\Lambda^V(\tau, \theta, \phi)$ is holomorphic using *a priori* estimates. This analyticity is in a smaller domain, but a Λ - independent domain.)

Proposition 7.1. *Assume QH, EL, and TR, with a fixed potential V. Then for fixed real θ and ϕ, the partition function $\mathfrak{Z}^V(\tau,\theta,\phi)$ is holomorphic in τ for all $\tau\in\mathbb{H}$. Furthermore, for fixed $\tau\in\mathbb{H}$ and fixed $\phi\in\mathbb{R}$, the function $\mathfrak{Z}^V(\tau,\theta,\phi)$ extends analytically in θ to a strip $|\Im(\theta)|<R$, where $R=R(\tau)$.*

Let $A=\Gamma e^{-i\theta J-i\sigma P}$. One can express the partition function $\mathfrak{Z}^V$ as

$$\begin{aligned}\mathfrak{Z}^V(\tau,\theta,\phi)&=\mathrm{Tr}_{\mathcal{H}}\left(Ae^{-\beta H}\right)=\mathrm{Tr}_{\mathcal{H}}\left(\Gamma e^{-i\theta J-i\tau\ell(H-P)/2+i\overline{\tau}\ell(H+P)/2}\right)\\&=\mathrm{Tr}_{\mathcal{H}}\left(\Gamma e^{-i\theta J-i\tau\ell\left(Q^2/2-P\right)+i\overline{\tau}\ell Q^2/2}\right),\end{aligned}\tag{7.1}$$

where $\overline{\tau}$ denotes the complex conjugate of τ. We have a representation similar to (7.1) for the approximating family of partition functions,

$$\mathfrak{Z}^V_\Lambda(\tau,\theta,\phi)=\mathrm{Tr}_{\mathcal{H}}\left(A\,e^{-\beta H_\Lambda}\right)=\mathrm{Tr}_{\mathcal{H}}(\Gamma e^{-i\theta J-i\tau\ell\left(Q^2_\Lambda/2-P\right)+i\overline{\tau}\ell Q^2_\Lambda/2}).\tag{7.2}$$

Lemma 7.2. *The approximating partition functions $\mathfrak{Z}^V_\Lambda(\sigma,\beta,\theta,\phi)$ are holomorphic in the following senses:*

(i) *Fix $\sigma\in\mathbb{R}$, $\theta\in\mathbb{R}$, and $\phi\in(0,\pi]$. Then $\mathfrak{Z}^V_\Lambda(\sigma,\beta,\theta,\phi)$ defined for $\beta>0$ is the boundary value of a holomorphic function of β extending to $i\beta\in\mathbb{H}$.*
(ii) *Fix $i\beta\in\mathbb{H}$, $\theta\in\mathbb{R}$, and $\phi\in(0,\pi]$. Then $\mathfrak{Z}^V_\Lambda(\sigma,\beta,\theta,\phi)$ extends analytically in σ into a strip around the real σ axis whose width is independent of Λ.*
(iii) *Fix $\sigma\in\mathbb{R}$, $i\beta\in\mathbb{H}$, and $\phi\in(0,\pi]$. Then $\mathfrak{Z}^V_\Lambda(\sigma,\beta,\theta,\phi)$ extends holomorphically in θ to a strip around the real θ axis, whose width is independent of Λ.*

Proof. Express the partition function $\mathfrak{Z}^V_\Lambda$ in terms of the real variables, $\mathfrak{Z}^V_\Lambda=\mathfrak{Z}^V_\Lambda(\sigma,\beta,\theta,\phi)=\mathrm{Tr}_{\mathcal{H}}\left(\Gamma e^{-i\theta J-i\sigma P-\beta H_j}\right)$. The uniform trace bound (5.3) ensures that $\mathfrak{Z}^V_\Lambda$ extends to a holomorphic function of β in the right half-plane. In order to establish part (ii), we use (5.21), combined with Lemma 5.2. Finally, to establish part (iii) of the lemma, we observe that J, P, and H_Λ are mutually commuting. Furthermore we use a bound on J in terms of $|P|$. In fact, using the explicit form of these operators, see [4], we conclude that for fixed $0<\phi$ there is a constant $M_3<\infty$ such that

$$\pm J\le M_3\,|P|.\tag{7.3}$$

It then follows from (5.21) that for constants M_1 and M_2, independent of Λ,

$$\pm J\le M_3\,(M_1\,H_\Lambda+M_2)\,.\tag{7.4}$$

We then apply Lemma 5.2 with θ replacing σ and J replacing P, to conclude that $\mathfrak{Z}^{\lambda V}_\Lambda(\tau,\theta,\phi)$ is real analytic in θ. The constants M_1, M_2, and M_3 do not depend on Λ, so there is a strip of uniform width about the real θ axis for which $\mathfrak{Z}^{\lambda V}_\Lambda$ is uniformly bounded and holomorphic. □

Lemma 7.3. *The approximate partition functions $\mathfrak{Z}^V_\Lambda(\sigma,\beta,\theta,\phi)$ satisfy the Cauchy–Riemann identity*

$$\frac{\partial\mathfrak{Z}^V_\Lambda}{\partial\sigma}+i\,\frac{\partial\mathfrak{Z}^V_\Lambda}{\partial\beta}=0,\tag{7.5}$$

for $\tau\in\mathbb{H}$. Therefore $\mathfrak{Z}^V_\Lambda$ is holomorphic for $\tau\in\mathbb{H}$.

Proof. By Lemma 7.2, the derivative of $\mathfrak{Z}_\Lambda^{\lambda V}(\sigma, \beta, \theta, \phi)$ with respect to β and σ exist. Differentiating the representation (7.2), and using the identity (4.14) yields

$$\frac{\partial \mathfrak{Z}_\Lambda^V}{\partial \sigma} + i\,\frac{\partial \mathfrak{Z}^V}{\partial \beta} = -i\,\mathrm{Tr}_{\mathcal{H}}\left(A\,\left(H_j + P\right) e^{-\beta H_j}\right) = -i\,\mathrm{Tr}_{\mathcal{H}}\left(A\,Q_\Lambda^2\,e^{-\beta H_\Lambda}\right). \quad (7.6)$$

Proposition 5.1 ensures that $Q_\Lambda(H_\Lambda + M_2)^{-1/2}$ is bounded, at least if we choose M_2 sufficiently large so $I \leq H_\Lambda + M_2$. But (5.3) ensures that $Q_\Lambda\,e^{-\beta H_\Lambda/2} = e^{-\beta H_\Lambda/2}\,Q_\Lambda$ is also bounded and trace class. As a consequence, we use cyclicity of the trace and $A\,Q_\Lambda = -Q_\Lambda\,A$ to give

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}}\left(A Q_\Lambda^2 e^{-\beta H_\Lambda}\right) &= \mathrm{Tr}_{\mathcal{H}}\left(A\,\left(e^{-\beta H_\Lambda/2}\,Q_\Lambda\right)\left(Q_\Lambda\,e^{-\beta H_j/2}\right)\right)\\ &= -\mathrm{Tr}_{\mathcal{H}}\left(\left(e^{-\beta H_\Lambda/2}\,Q_\Lambda\right)\,A\,\left(Q_\Lambda\,e^{-\beta H_\Lambda/2}\right)\right)\\ &= -\mathrm{Tr}_{\mathcal{H}}\left(A\,Q_\Lambda\,e^{-\beta H_\Lambda/2}\,Q_\Lambda\,e^{-\beta H_\Lambda/2}\right)\\ &= -\mathrm{Tr}_{\mathcal{H}}\left(A Q_\Lambda^2 e^{-\beta H_\Lambda}\right) = 0,\end{aligned} \quad (7.7)$$

completing the proof of analyticity in $\tau \in \mathbb{H}$. □

8. Evaluation

We verify the representation for the elliptic genus in the case that the potential V is zero.

Proposition 8.1. *Choose $\Omega_i \in (0, \frac{1}{2}]$ for $1 \leq i \leq n$. Take $V = 0$ and assume TR and NC. Then the partition function $\mathfrak{Z}^0$ is given by (2.4).*

Proof. Define

$$\Omega_i^f(k) = \begin{cases}\Omega_i, & \text{if } 0 < k\\ 1 - \Omega_i, & \text{if } k < 0\end{cases}, \quad (8.1)$$

and the functions

$$\gamma_{\pm,i}^b(\pm k) = e^{\mp i\theta\Omega_i - \beta|k|}, \qquad \text{and } \gamma_{\pm,i}^f(\pm k) = e^{\mp i\theta\Omega_i^f(\pm k) - \beta|k|}. \quad (8.2)$$

The momenta range over the following lattices,

$$K_i^b = \{k : \ell k \in 2\pi\mathbb{Z} - \Omega_i\phi\}, \qquad \text{and } K_{\pm,i}^f = \{k : \ell k \in 2\pi\mathbb{Z} - \Omega_i^f(\pm k)\phi\}. \quad (8.3)$$

We require that $0 < \phi \leq 2\pi$ and $0 < \Omega_i,\ 1 - \Omega_1 < 1$, so zero is not an allowed momentum,

$$0 \notin K_i^b,\ \ K_{+,i}^f,\ \ K_{-,i}^f. \quad (8.4)$$

In case $V = 0$, the partition function factors into a product of a fermionic free-field part and a bosonic free-field part. We calculated the free bosonic and fermionic partition functions in Theorems 2.2.1 and 5.4.1 of [4], yielding

$$\mathfrak{Z}^0 = y^{\hat{c}/2}\prod_{i=i}^{n}\left(\prod_{k\in K_i^b}\prod_{k'\in K_{+,i}^f}\prod_{k''\in K_{-,i}^f}\frac{\left(1-\gamma_{+,i}^f(k')\right)\left(1-\gamma_{-,i}^f(-k'')\right)}{|1-\gamma_{+,i}^b(k)|^2}\right). \quad (8.5)$$

The overall factor $y^{\hat{c}/2}$ arises from the normalization constant $\hat{c}/2$ in (1.13).

Split each product into terms indexed by $n \in \mathbb{Z}$, and separate the terms with positive, negative, and zero n. Note that $\gamma^b_{+,i}(k) = \gamma^b_{-,i}(-k)^*$. (The $\gamma^f_{\pm,i}$ satisfy such a relation only when $\Omega_i = 1/2$.) For $k = \left(2\pi n - \chi^{b,f}_{\pm,i}\right)/\ell$, and $n \in \mathbb{Z}$ the functions $\gamma^{b,f}_{\pm,i}(\pm k)$ take the following values:

	$\gamma^b_{+,i}(k)$	$\gamma^b_{-,i}(-k)$	$\gamma^f_{+,i}(k)$	$\gamma^f_{-,i}(-k)$
$n = 0$	$(z/y)^{\Omega_i}$	$(y\overline{z})^{\Omega_i}$	$(z/y)^{1-\Omega_i}$	$(y\overline{z})^{\Omega_i}$
$n > 0$	$\overline{q}^n\,(1/y\overline{z})^{\Omega_i}$	$q^n\,(y/z)^{\Omega_i}$	$\overline{q}^n\,(1/y\overline{z})^{\Omega_i}$	$q^n\,(y/z)^{1-\Omega_i}$
$n < 0$	$q^{\lvert n\rvert}\,(z/y)^{\Omega_i}$	$\overline{q}^{\lvert n\rvert}\,(y\overline{z})^{\Omega_i}$	$q^{\lvert n\rvert}\,(z/y)^{1-\Omega_i}$	$\overline{q}^{\lvert n\rvert}\,(y\overline{z})^{\Omega_i}$

Therefore (8.5) equals the product of ratios made from these 12 terms, with factors $1-\gamma^f$ in the numerator and factors $1-\gamma^b$ in the denominator. Group the terms depending on $\overline{q}$ near each other, to obtain

$$\begin{aligned}\mathfrak{Z}^0 = y^{\hat{c}/2}\prod_{i=i}^{n}\Bigg\{&\frac{(1-(z/y)^{1-\Omega_i})(1-(y\overline{z})^{\Omega_i})}{(1-(z/y)^{\Omega_i})(1-(y\overline{z})^{\Omega_i})}\\ \times\prod_{n=1}^{\infty}&\frac{\left(1-\overline{q}^n(1/y\overline{z})^{\Omega_i}\right)\left(1-\overline{q}^n(y\overline{z})^{\Omega_i}\right)\left(1-q^n(z/y)^{1-\Omega_i}\right)\left(1-q^n(y/z)^{1-\Omega_i}\right)}{\left(1-\overline{q}^n(1/y\overline{z})^{\Omega_i}\right)\left(1-\overline{q}^n(y\overline{z})^{\Omega_i}\right)\left(1-q^n(z/y)^{\Omega_i}\right)\left(1-q^n(y/z)^{\Omega_i}\right)}\Bigg\}.\end{aligned} \tag{8.6}$$

Using the definition (1.16), the product (8.6) is

$$\begin{aligned}\mathfrak{Z}^0 &= z^{\hat{c}/2}\prod_{i=i}^{n}\frac{\vartheta_1(-\overline{\tau},\Omega_i\,(\phi\overline{\tau}+\theta))\,\vartheta_1(\tau,(1-\Omega_i)\,(\phi\tau-\theta))}{\vartheta_1(-\overline{\tau},\Omega_i\,(\phi\overline{\tau}+\theta))\,\vartheta_1(\tau,\Omega_i\,(\phi\tau-\theta))}\\ &= z^{\hat{c}/2}\prod_{i=i}^{n}\frac{\vartheta_1(\tau,(1-\Omega_i)\,(\theta-\phi\tau))}{\vartheta_1(\tau,\Omega_i\,(\theta-\phi\tau))}.\end{aligned} \tag{8.7}$$

The theta functions depending on $\overline{\tau}$ occur in both the numerator and the denominator. We also use here the fact that the function ϑ_1 is odd in the second variable. This completes the evaluation of $\mathfrak{Z}^0$, and it also completes the proof of Corollary 2.2. □

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